Unit 1 - Vector Space

1.1 Def^n of vector space and simple example

1.2 Th^m - In any vector space $V(F)$ the following results hold

i) $0 \cdot x = 0$ ii) $\alpha \cdot 0 = 0$

iii) $(-\alpha) \cdot x = -(\alpha x) = \alpha(-x)$

iv) $(\alpha - \beta) x = \alpha x - \beta x$

1.3 Def^n of sub space

1.4 Th^m - A necessary and sufficient condition for a non-empty subset of a vector space $V(F)$ to be a subspace is that w is closed under addition and scalar multiplication

1.5 Th^m - A non-empty subset W of a vector space $V(F)$ is a subspace of V iff $\alpha x + \beta y \in W$ for $\alpha, \beta \in F$, $x, y \in W$

1.6 Def^n of sum of subspaces, direct sum, quotient space, homomorphism of vector space and examples

1.7 Th^m - Under a homomorphism $T: V \to U$

i) $T(0) = 0$, ii) $T(-x) = -T(x)$

1.8 Def^n of Kernel and range of homomorphism

1.9 Th^m - Let $T: V \to U$ be a homomorphism then $\operatorname{Ker} T$ is a sub-space of V

1.10 Th^m - Let $T: V \to U$ be a homomorphism then $\operatorname{Ker} T = \{0\}$ iff T is one to one

1.11 Th^m - Let $T: V \to U$ be a linear transformation (L.T.) then range of T is a subspace of U.

1.12 Th^m - Let W be subspace of V then their exist an onto linear transformation $\theta: V \to V/W$ such that $\operatorname{Ker} \theta = W$

1.13 Def^n of linear span

1.14 Th^m - $L(S)$ is the smallest subspace of V containing S

1.15 Th^m - If W is subspace of V then $L(W) = W$ and conversly.

1.16 Def^n of finite dimensional vector space (F.D.V.S.), linear dependence and independence basis of vector space and examples

1.17 Th^m - If $S = \{v_1, v_2, v_3, \ldots v_n\}$ is a basis of V then every element of V can be expressed uniquely as a linear combination of $v_1, v_2, v_3, \ldots, v_n$

1.18 Th^m - Suppose S is a finite subspace of vector space V such that $V = L(S)$ then their exist a subset of S which is a basis of V

1.19 Def^n of F.D.V.S.

1.20 Th^m - If V is a F.D.V.S. and $\{v_1, v_2, \ldots v_r\}$ is a linearly independent [L.I.] subset of V then it can be extended to form a basis of V

1.21 Th^m - If $\dim V = n$ and $S = \{v_1, v_2, \ldots v_n\}$ spans V then S is a basis of V

1.22 Th^m - If $\dim V = n$ and $S = \{v_1, v_2, \ldots v_n\}$ is a linearly independent subset of V then S is a basis of V

Unit 2 - Inner Product Spaces

2.1 Def^n of inner product space, norm of a vector and examples

2.2 Th^m - Cauchy - schwarz inequality
let V be an inner product space then
$|(u,v)| \leq \|u\|\|v\|$, $u, v \in V$

2.3 Th^m - Triangle Inequality
Let V be an inner product space then norm of $||u+v|| \leq ||u|| + ||v|| \quad \forall \; u, v \in V$

2.4 Th^m - Parallelogram Law
Let V be an inner product space then
$||u+v||^2 + ||u-v||^2 = 2\,(||u||^2 + ||v||^2) \;\; \forall \; u, v \in V$

2.5 Def^n of orthogonal vectors and orthonormal sets

2.6 Th^m - Let S be an orthogonal set of non-zero vectors in an inner product space V then S is a linearly independent space

2.7 Gram - schmidt orthogonalization space

2.7.1 Th^m - Let V be a non-trivial inner product space of dimension n then V has an orthonormal basis

2.7.2 examples

Unit 3 - Linear Transformation

3.1 Def^n of L.T.; rank and nullity and examples

3.2 Th^m - A L.T. $T: V \to V$ is one-to-one iff T is onto.

3.3 Th^m - Let V and W be two vector spaces over F
Let $\{v_1, v_2, \ldots \ldots v_n\}$ be a basis of V and $w_1, w_2, \ldots \ldots w_n$ be any vectors in W (not essentially distinct) then their exist a unique L.T. $T: V \to W$ such that $T(v_i) = w_i \quad i = 1, 2, \ldots n$

3.4 Th^m - Sylvester's Law
Suppose V and W are F.D.V.S. over a field F
Let $T: V \to W$ be a L.T. then rank T + nullity T = dim V

3.5 Th^m - If $T: V \to V$ be a L.T. show that the following statements are equivalent.

i) range $T \cap \text{Ker } T = \{0\}$
ii) If $T(T(v)) = 0$ then $T(v) = 0$, $v \in V$

3.6 Def^n of sum and product of L.T., linear operator, linear functional and examples.

3.7 Th^m - Let T, T_1, T_2 be linear operators on V and Let $I : V \to V$ be the identity mapping $I(v) = v \quad \forall \; v$ then
i) $IT = TI = T$
ii) $T \cdot (T_1 + T_2) = TT_1 + TT_2$, $(T_1 + T_2)T = T_1T + T_2T$
iii) $\alpha \cdot (T_1 T_2) = (\alpha T_1) T_2 = T_1 (\alpha T_2)$, $\alpha \in F$
iv) $T_1 (T_1 T_2) = (T_1 T_2) T_1$

3.8 Def^n of invertible L.T. and examples

3.9 Th^m - A L.T. $T : V \to W$ is a non-singular iff T carries each L.I. subset of V onto a L.I. subset of W

3.10 Th^m - Let $T : V \to W$ be a L.T. where V and W are F.D.V.S. with same dimension then the following are equivalent.
i) T is invertible. ii) T is non-singular.
iii) T is onto

3.11 Th^m - Let $T : V \to W$ and $S : W \to V$ be two L.T. then
i) If S and T are one-one onto then ST is one-one onto & $(ST)^{-1} = T^{-1} \cdot S^{-1}$
ii) If ST is one-one then T is one-one
iii) If ST is onto then S is onto.

3.12 Def^n of matrix of L.T. and examples

3.12.1 Th^m : Hom $(U, V) \cong M_{m \times n}(F)$

3.13 Def^n of dual space

3.13.1 Th^m : Let V be n dimensional vector space over a field F then the dimension of dual space of V over F is n

Unit 4 : Eigen values and Eigen vectors

4.1 Def^n of eigen values, eigen vectors, eigen space

4.2 Let A be a square matrix of order n if λ is an eigen value of A then the set of all eigen vectors of A corresponding to $\cancel{x}$ λ together with vector forms a subspace of n-dimensional unitory space

4.3 Th^m - Let A be a square matrix of order n having K distinct eigen values $\lambda_1, \lambda_2, \ldots \lambda_k$ Let v_i be an eigen vector corresponding to the eigen value λ_i $i = 1, 2 \ldots k$ then the set $\{v_1, v_2, \ldots v_k\}$ is linearly independent.

4.4 Th^m - Let A be a square matrix of order n having n distinct eigen values $\lambda_1, \lambda_2, \ldots \lambda_n$. Let v_i be an eigen vector corresponding to the eigen value λ_i $i = 1, 2, \ldots n$ the the set $(v_1, v_2, \ldots v_n)$ is basis for the domain space of A. The matrix of the linear transformation w.r.t. the basis

$$\begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

4.5 Examples on application of 4.4

Vector Spaces

Def^n - Vector Space

Let $\langle V, + \rangle$ be an abelian group and $\langle F + \cdot \rangle$ be a field. Define a function $\cdot$ (scalar multiplication) $\cdot : F \times V \to V$ such that for all $\alpha \in F$, $v \in V$, $\alpha \cdot v \in V$ then V is said to form a vector space over F if for all $x, y \in V$, $\alpha, \beta \in F$ The following holds

1) $(\alpha + \beta) x = \alpha x + \beta x$
2) $(\alpha)(x+y) = \alpha x + \alpha y$
3) $(\alpha \beta) x = \alpha(\beta x) = \beta(\alpha x)$
4) $1 \cdot x = x$, 1 is unity of F

It is denoted by $V(F)$

Note -
Members of F are called scalars and those of V are called vectors

Examples -

1) $R(R)$, $C(R)$, $C(C)$ are vector space. But $R(C)$ is not a vector space.
since $\langle R + \rangle$ is an abelian group but for $2 \in R$, $2+3i \in C$

$(2+3i) \cdot 2 = 4 + 6i \notin R$

$\therefore$ R is not closed w.r.t. scalar multiplication

$\therefore$ $R(C)$ is not a vector space.

2) Let $\langle F + \cdot \rangle$ be a field. Let $\forall$
$V = \{ (\alpha_1, \alpha_2) / \alpha_1, \alpha_2 \in F\} = F \times F = F^2$
Define $+$ and $\cdot$ (scalar multiplication) by
$(\alpha_1, \alpha_2) + (\beta_1, \beta_2) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2)$

$\alpha (\alpha_1, \alpha_2) = (\alpha\alpha_1, \alpha\alpha_2)$ then V is vector space over F.

$\rightarrow$ Let $\langle F, +, \cdot \rangle$ be a field.
Let $V = \{(\alpha_1, \alpha_2) / \alpha_1, \alpha_2 \in F\} = F \times F = F^2$

Prove that $\langle V, + \rangle$ is an abelian group.

1) Let (α_1, α_2) & $(\beta_1, \beta_2) \in V$
$\therefore (\alpha_1, \alpha_2) + (\beta_1, \beta_2) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2) \in V$

2) Let $u = (\alpha_1, \alpha_2)$, $v = (\beta_1, \beta_2)$, $\omega = (\gamma_1, \gamma_2) \in V$

$$u + (v + \omega) = (\alpha_1, \alpha_2) + ((\beta_1, \beta_2) + (\gamma_1, \gamma_2))$$

$$= (\alpha_1, \alpha_2) + ((\beta_1 + \gamma_1) + (\beta_2 + \gamma_2))$$

$$= (\alpha_1 + (\beta_1 + \gamma_1), \alpha_2 + (\beta_2 + \gamma_2))$$

$$= ((\alpha_1 + \beta_1) + \gamma_1, (\alpha_2 + \beta_2) + \gamma_2)$$

$$= (\alpha_1 + \beta_1, \alpha_2 + \beta_2) + (\gamma_1, \gamma_2)$$

$$= (u + v) + \omega$$

3) For $u = (\alpha_1, \alpha_2) \in V$ then their exist $\bar{0} = (0,0) \in V$ such that,
$u + \bar{0} = u = \bar{0} + u$
$\therefore$ $\bar{0} = (0, 0)$ is a zero vector in V

4) For $u = (\alpha_1, \alpha_2) \in V$ their exist $-u = (-\alpha_1, -\alpha_2) \in V$ such that
$u + (-u) = \bar{0} = (-u) + u$
$\therefore$ inverse exist.

5) For $u = (\alpha_1, \alpha_2)$ and $(\beta_1, \beta_2) \in V$ $\therefore$

$u + v = (\alpha_1, \alpha_2) + (\beta_1, \beta_2)$

$= (\alpha_1 + \beta_1, \alpha_2 + \beta_2)$

$= (\beta_1 + \alpha_1, \beta_2 + \alpha_2)$

$= (\beta_1, \beta_2) + (\alpha_1, \alpha_2)$

$u + v = v + u \quad \forall \; u, v \in V$

$\therefore$ $\langle V + \rangle$ is an abelian group.

6) since $u = (\alpha_1, \alpha_2) \in V$, $\alpha \in F$
$\therefore$ $\alpha \cdot u = \alpha(\alpha_1, \alpha_2)$
$= (\alpha\alpha_1, \alpha\alpha_2) \in V$

$\therefore$ V is closed w.r.t. scalar multiplication.

7) Let $\alpha, \beta \in F$, $u = (\alpha_1, \alpha_2) \in V$
$(\alpha + \beta) u = (\alpha + \beta)(\alpha_1, \alpha_2)$
$= ((\alpha + \beta)\alpha_1, (\alpha + \beta)\alpha_2)$ Def^n of scalar multiplication

$$= (\alpha\alpha_1 + \beta\alpha_1 , \alpha\alpha_2 + \beta\alpha_2)$$ distributive law

$$= (\alpha\alpha_1 , \alpha\alpha_2) + (\beta\alpha_1 + \beta\alpha_2)$$ Def^n

$$= \alpha(\alpha_1 , \alpha_2) + \beta(\alpha_1 , \alpha_2)$$
$$= \alpha u + \beta u$$

8) $\alpha(x+y) = \alpha x + \alpha y$

Let $\alpha \in F$, $u = (\alpha_1 , \alpha_2)$, $v = (\beta_1 , \beta_2) \in V$

$$\alpha(u+v) = \alpha((\alpha_1 , \alpha_2) + (\beta_1 ; \beta_2))$$

$$= \alpha(\alpha_1 + \beta_1 , \alpha_2 + \beta_2)$$

$$= (\alpha(\alpha_1 + \beta_1) , \alpha(\alpha_2 + \beta_2))$$

$$= (\alpha\alpha_1 + \alpha\beta_1 , \alpha\alpha_2 + \alpha\beta_2)$$

$$= (\alpha\alpha_1 , \alpha\alpha_2) + (\alpha\beta_1 ; \alpha\beta_2)$$

$$= \alpha(\alpha_1 , \alpha_2) + \alpha(\beta_1 , \beta_2)$$
$$= \alpha u + \alpha v$$

9) $(\alpha\beta)x = \alpha(\beta x)$

Let $\alpha, \beta \in F$, $u = (\alpha_1 , \alpha_2) \in V$

$$(\alpha\beta)u = (\alpha\beta)(\alpha_1 + \alpha_2)$$

$$= ((\alpha\beta)\alpha_1 , (\alpha\beta)\alpha_2)$$

$$= (\alpha(\beta\alpha_1) , \alpha(\beta\alpha_2))$$

$= \alpha(\beta\alpha_1, \beta\alpha_2)$

$= \alpha(\beta(\alpha_1, \alpha_2))$

$= \alpha(\beta u)$

10) $1 \cdot x = x$

$u = (\alpha_1, \alpha_2) \in V$

$1 \cdot u = 1 \cdot (\alpha_1, \alpha_2)$

$= (\alpha_1, \alpha_2)$

$= u.$

Note -
In general we can take n- tuples
$V = F^n = \{(\alpha_1, \alpha_2, \ldots \alpha_n) / \alpha_1, \alpha_2 \ldots \alpha_n \in F\}$

Define $+$ and $\cdot$ as

$u + v = (\alpha_1, \alpha_2, \ldots \alpha_n) + (\beta_1, \beta_2, \ldots \beta_n)$

$= (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_n + \beta_n)$

$\alpha u = \alpha(\alpha_1, \alpha_2, \ldots, \alpha_n)$

$= (\alpha\alpha_1, \alpha\alpha_2, \ldots, \alpha\alpha_n)$

then V is a vector space over F w.r.t. usual vector addition and scalar multiplication.

3) Check whether $R(Q)$ and $Q(R)$ are vector space or not.

→ $R(Q)$ is a vector space.
$Q(R)$ is not a vector space.

4) Let V be the set of all real valued continuous functions defined on $[0,1]$ then V forms a vector space over the field R of reals under addition and scalar multiplication defined as

$$(f+g)(x) = f(x) + g(x)$$
$$(\alpha f)(x) = \alpha f(x) \qquad \forall\, x \in [0,1],\ g, f \in V,\ \alpha \in R$$

→ Now clearly as $f+g$ and $\alpha \cdot f$ are again real valued continuous function defined on $[0,1]$

∴ V is closed w.r.t. vector addition and scalar multiplication.

1) Associativity

For any $x \in [0,1]$ we have

Let $f, g, h \in V$

$$[(f+g)+h](x) = [(f+g)(x) + h(x)]$$
$$= [(f(x) + g(x)) + h(x)]$$
$$= [f(x) + g(x) + h(x)]$$
$$= [f + (g+h)](x)$$

$$(f+g)+h = f + (g+h) \qquad f, g, h \in V.$$

2) For any $x \in [0,1]$ we define a function $\overline{0}$ from $[0,1]$ to R by $\overline{0} \cdot x = 0$.

Then for any $f \in V$

$$(f+\bar{0})(x) = f(x) + \bar{0}(x)$$
$$= f(x) + 0$$
$$= f(x)$$

$\therefore \; f + \bar{0} = f \qquad \forall\, f \in V$

3) For any $f \in V$ we define ~~$-f$ from $[0\ 1]$ to~~ $-f : [0\ 1] \to R$ as $(-f)(x) = -f(x)$. then we have,

$$[f + (-f)](x) = f(x) + (-f)(x)$$
$$= f(x) - f(x)$$
$$= 0$$
$$= \bar{0}(x)$$

4) Commutativity

For any $x \in [0.1]$ Let $f, g \in V$

$$(f+g)(x) = f(x) + g(x)$$
$$= g(x) + f(x)$$
$$= \{g+f\}(x)$$

$\Rightarrow \quad f+g = g+f$

5) For any scalars α, β and $f \in V$

consider $[(\alpha+\beta)f](x) = (\alpha+\beta) f(x)$
$$= \alpha f(x) + \beta f(x)$$
$$= (\alpha f)(x) + (\beta f)(x)$$
$$= (\alpha f + \beta f)x$$

$\Rightarrow \quad (\alpha+\beta)f = \alpha f + \beta f$

6) For any scalar α and $f, g \in V$

$$[\alpha[(f+g)](x) = \alpha(f+g)(x)$$

$$= \alpha[f(x)+g(x)]$$

$$= \alpha f(x) + \alpha g(x)$$

$$= (\alpha f)(x) + (\alpha g)(x)$$

$$= (\alpha f + \alpha g)(x)$$

$$\Rightarrow \alpha(f+g) = \alpha f + \alpha g$$

7) For any scalars α, β and any $f \in V$

$$[(\alpha\beta)f](x) = (\alpha\beta)f(x)$$

$$= \alpha(\beta f(x))$$

$$= \alpha[(\beta f)(x)]$$

$$= [\alpha(\beta f)](x)$$

$$\Rightarrow (\alpha\beta)f = \alpha(\beta f)$$

8) For any $f \in V$ and any $x \in [0,1]$

$$[1 \cdot f](x) = 1 \cdot f(x)$$

$$= f(x)$$

$$\Rightarrow 1 \cdot f = f.$$

Thus V is a vector space over R

5) Let P be the set of all polynomials over a field F then P forms a vector space under additio[n] and scalar multiplication.

6) Let $M_{m \times n}(F)$ = The set of all $m \times n$ matrices with entries from field F forms a vector space under matrix addition & scalar multiplication of matrices.

Theorem

In any vector space V(F) the following results holds

i) $0 \cdot x = 0$, ii) $\alpha \cdot 0 = 0$

iii) $(-\alpha) x = -(\alpha x) = \alpha(-x)$

iv) $(\alpha - \beta) x = \alpha x - \beta x$

→ Let $0 \in F$ and $x \in V$

Now $0 + 0 = 0$

$\Rightarrow (0+0) x = 0 \cdot x$

$\Rightarrow 0 \cdot x + 0 \cdot x = 0 \cdot x$ since V is vector space

$\Rightarrow 0 \cdot x + 0 \cdot x = 0 \cdot x + 0$

$0 \cdot x = 0$

For $\alpha \in F$ and $0 \in V$

Now $0 + 0 = 0$

$\alpha(0+0) = \alpha \cdot 0$

$\alpha \cdot 0 + \alpha \cdot 0 = \alpha \cdot 0$

$\alpha \cdot 0 + \alpha \cdot 0 = \alpha \cdot 0 + 0$

$\alpha \cdot 0 = 0$

iii) Let $\alpha \in F$, $x \in V$, $-\alpha \in F$

$$\alpha + (-\alpha) = 0.$$

$$[\alpha + (-\alpha)] \cdot x = 0 \cdot x$$

$$\alpha x + (-\alpha) x = 0 \quad \because 0 \cdot x = 0$$

$||^{ly}$

$$(-\alpha) x + \alpha x = 0.$$

$\therefore$ $(-\alpha) \cdot x$ is an additive inverse of αx

$\therefore$ $(-\alpha) x = -(\alpha x)$

Let $\alpha \in F$, $x \in V$, $-x \in V$

consider,

$$x + (-x) = 0.$$

$$\alpha [x + (-x)] = \alpha \cdot 0$$

$$\alpha x + \alpha(-x) = 0 \quad \because V \text{ is vector space}$$

$||^{ly}$

$$\alpha(-x) + \alpha x = 0.$$

$\alpha(-x)$ is an additive inverse of αx.

$\therefore$ $\alpha(-x) = -(\alpha x)$

$\therefore$ $(-\alpha) x = -(\alpha x) = \alpha(-x)$

iv) Let $\alpha, \beta \in F$, $x \in V$

$$\therefore \; \cancel{(\alpha)} \; (\alpha - \beta)(x) = (\alpha + (-\beta)) x$$
$$= \alpha x + (-\beta) x$$
$$= \alpha x - \beta x$$

* Subspace

Def^n - A non-empty subset W of a vector space V(F) is said to form a subspace of V if W forms a vector space over F under the operations in V

* Theorem

A necessary and sufficient condition for non-empty subset W of a vector space V(F) to be subspace is that W is closed under addition and scalar multiplication.

i.e. $x, y \in W \Rightarrow x+y \in W$

$\alpha \in F,\ x \in W \Rightarrow \alpha \cdot x \in W$

→ If W is a subspace of V(F) then result follows by def^n i.e. W is closed w.r.t. vector addition and scalar multiplication.

Conversly, let W is closed under addition and scalar multiplication.

To prove that W is a subspace of V i.e. prove that W is a vector space under same operations in V.

Now we prove that W is a subgroup of V w.r.t. '+'.

Let $x, y \in W$, $-1 \in F$

$\therefore -y = (-1)y \in W$

$\therefore x-y = x+(-y) \in W$

$\therefore x, y \in W \Rightarrow x-y \in W$

$\therefore \langle W, + \rangle$ is a subgroup of $\langle V + \rangle$

Remaining conditions in the defination followed trivially.

(1)* Theorem

A non-empty subset W of a vector space V(F) is a subspace of V iff $\alpha x + \beta y \in W$ $\forall$ $x, y \in W$ & $\alpha, \beta \in F$.

→ Let V(F) be a vector space.
Let W be a non-empty subset of V
If W is a subspace of V then W is closed w.r.t. addition and scalar multiplication

$x, y \in W \Rightarrow x+y \in W$

$\alpha \in F$, $x \in W \Rightarrow \alpha x \in W$

Let $x, y \in W$, $\alpha, \beta \in F$

$\alpha x \in W$, $\beta y \in W$ — By Def^n

$\Rightarrow \alpha x + \beta y \in W$ — By Def^n

Conversly let the given condition hold in W
i.e $\alpha x + \beta y \in W$ $\forall$ $x, y \in W$ & $\alpha, \beta \in F$

For $\alpha = \beta = 1$
let $x, y \in W$

$\therefore$ $x + y = 1 \cdot x + 1 \cdot y \in W$ $\forall$ $x, y \in W$

$\therefore$ W is closed w.r.t. vector addition.

For $\beta = 0$, $\alpha \in F$; $x, y \in W$

$\therefore \alpha x = \alpha x + 0 \cdot y \in W$

$\Rightarrow \alpha x \in W \quad \forall\ x \in W, \alpha \in F$.

$\therefore$ W is closed w.r.t. scalar multiplication.

We know that a nonempty subset W of a vector space V(F) is a subspace iff W is closed w.r.t. vector addition and scalar multiplication

$\therefore$ W is subspace of V(F).

Examples -

1) Consider the vector space $R^2(R)$ then show that ~~W(1)~~ $W_1 = \{(a,0) / a \in R\}$ and $W_2 = \{(0,b) / b \in R\}$ are subspaces of $R^2(R)$

$\rightarrow$ Let, $R^2 = \{(x,y) / x, y \in R\}$ is a vector space w.r.t. usual vector addition & scalar multiplication

$W_1 = \{(a,0) / a \in R\}$

By def^n $(0,0) \in W_1 \subseteq R^2$

$\therefore$ W_1 is non-empty subset of R^2. —— ①

Let $u, v \in W_1$

$\therefore u = (a_1, 0)$ and $v = (a_2, 0)$

$\therefore u + v = (a_1, 0) + (a_2, 0)$

$= (a_1 + a_2, 0) \in W_1$

$\therefore u, v \in W_1 \Rightarrow u + v \in W_1$ —— ②

Let $\alpha \in R$, $u = (a, 0) \in W_1$

$\therefore \alpha u = \alpha(a, 0) = (\alpha a, 0) \in W_1$

$\therefore \alpha \in R, u \in W_1 \Rightarrow \alpha u \in W_1$ —— ③

$\therefore$ From ①, ②, ③ W_1 is a subspace of $R^2(R)$.

Let $u_1 = (x_1, y_1, z_1)$ and $u_2 = (x_2, y_2, z_2) \in W$

$\therefore 2x_1 + 3y_1 + 5z_1 \quad , \quad 2x_2 + 3y_2 + 5z_2 = 0.$

$\therefore u_1 + u_2 = (x_1, y_1, z_1) + (x_2, y_2, z_2)$
$= \underline{(x_1 + x_2, y_1 + y_2, z_1 + z_2)}$

$2(x_1 + x_2) + 3(y_1 + y_2) + 5(z_1 + z_2)$
$= (2x_1 + 3y_1 + z_1) + (2x_2 + 3y_2 + 5z_2)$
$= 0 + 0$
$= 0.$

$\therefore u_1 + u_2 \in W \quad \forall \; u_1, u_2 \in W$ — ②

Let α be a scalar and $u = (x, y, z) \in W$
$\therefore 2x + 3y + 5z = 0.$

$\alpha u = \alpha(x, y, z) = (\alpha x, \alpha y, \alpha z)$

$\therefore 2(\alpha x) + 3(\alpha y) + 5(\alpha z) = \alpha(2x + 3y + 5z)$
$= \alpha \cdot 0$
$= 0.$

$\therefore \alpha u \in W \quad \forall$ scalar α & $u \in W$ — ③

$\therefore$ From ①, ②, ③
W is subspace of R^3.

3) Consider verctor space $V = R^3$. Let $W = \{(x, y, 0) \in R^3 / x, y \in R\}$. Show that W is subspace of R^3.

→ Let $R^3 = \{(x, y, z) / x, y, z \in R\}$ is a vector space w.r.t. usual addition and scalar multiplication.

Let $W = \{(x, y, 0) \in R^3 / x, y \in R\}$

By def^n

$(0, 0, 0) \in W \subseteq R^3$

$\therefore$ W is a non-empty subset of R^3 — ①.

Let $u_1, u_2 \in W$, $u_1 = (x_1, y_1, 0)$, $u_2(x_2, y_2, 0)$

$\therefore u_1 + u_2 = (x_1, y_1, 0) + (x_2, y_2, 0)$
$= (x_1 + x_2, y_1 + y_2, 0)$

$\therefore u_1 + u_2 \in W \quad \forall \quad u_1, u_2 \in W$ — ②

Let α be a scalar and $u = (x, y, 0) \in W$

$\therefore \alpha u = \alpha(x, y, 0) = (\alpha x, \alpha y, 0) \in W$

$\therefore \alpha u \in W \quad \forall \quad \alpha$ is scalar, $u \in W$ — ③

From ①, ②, ③

W is subspace of R^3.

4) Determine whether the following sets are subspace of R^2 or not.

1) $W = \{(x, y) \in R^2 / y = x^2\}$

→ Let $R^2 = \{(x, y) / x, y \in R\}$ be a vector space.

Let $W = \{(x, y) \in R^2 / y = x^2\}$

By def^n $(0, 0) \in W \subseteq R^2$.

$\therefore$ W is non-empty subset of R^2.

Let $u_1, u_2 \in W$ $u_1 = (x_1, y_1)$, $u_2 = (x_2, y_2) \in W$

$y_1 = x_1^2$, $y_2 = x_2^2$

$\therefore u_1 + u_2 = (x_1, y_1) + (x_2, y_2)$
$= (x_1 + x_2, y_1 + y_2)$

$\therefore y_1 + y_2 = x_1^2 + x_2^2 \neq (x_1 + x_2)^2$

$\therefore y_1 + y_2 \notin W$

$\therefore (1, 1), (2, 4) \in W$

$\therefore (1, 1) + (2, 4) \notin W$

$\therefore$ W is not closed w.r.t. addion.

$\therefore$ W is not subspace of R^3.

5) Determine whether the following sets are subspace of R^3 or not.

1) $W = \{(x, y, z) \in R^3 / y = 2x, z = 3x, x \in R\}$

2) $W \; \{(x, 2x, 3x) \cancel{\in R^3} / x \in R\}$

→1) let $R^3 = \{(x,y,z) \in R^3 / x, y, z \in R\}$

let $W = \{(x,y,z) \in R^3 / y = 2x, z = 3x, x\}$

By defn $(0,0,0) \in R^3$

$\therefore (0,0,0) \in W$.

$\therefore W$ is subset of R^3 — ①.

$u_1, u_2 \in W$, $u_1 = (x_1, 2x_1, 3x_1)$, $u_2(x_1, 2x_2, 3x_3)$

$\therefore u_1 + u_2 = (x_1, 2x_1, 3x_1) + (x_1 + 2x_2, 3x_3)$

$= (x_1 + x_2) + 2(x_1 + x_2) + 3(x_1 + x_2 + x_3) \in W$.

$\therefore u_1, u_2 \in W$, $u_1 + u_2 \in W$ $\forall$ $u_1, u_2 \in W$ — ②.

Let α be a scalar, $u \in W$ then, $u = (x, 2x, 3x)$

$\alpha u = \alpha(x, 2x, 3x) = (\alpha x, 2\alpha x, 3\alpha x) \in W$

$\therefore \alpha u \in W$

$\therefore$ W is subspace w.r.t. R^3

→ 2) $W = \{(x,y,z) \in R^3 / x = y + z\}$

let $W = \{(x,y,z) \in R^3 / x = y + z\}$.

By defn $(0,0,0) \in R^3$

$\therefore (0,0,0) \in W \subseteq R^3$

$\therefore$ W is non-empty subset of R^3. —— ①

Let $u_1, u_2 \in W$, $u_1 = (x_1, y_1, z_1)$, $u_2(x_2, y_2, z_2)$
$x_1 = y_1 + z_1$, $x_2 = y_2 + z_2$
$\therefore u_1 + u_2 = (x_1, y_1, z_1) + (x_2, y_2, z_2)$
$= (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$\therefore (x_1 + x_2) = (y_1 + z_1) + (y_2 + z_2)$
$= (y_1 + y_2) + (z_1 + z_2)$

$\therefore u_1 + u_2 \in W \quad \forall \; u_1, u_2 \in W$

Let α be a scalar and $u = (x, y, z)$.
$\therefore x = y + z$.

$\alpha u = \alpha \cdot (x, y, z) = (\alpha x, \alpha y, \alpha z)$

$\therefore \alpha x = \alpha(y + z)$
$= \alpha y + \alpha z$

$\therefore \alpha u \in W \quad \forall$ scalars α & $u \in W$.

$\therefore$ W is subspace of R^3

7) $W = \{(x, y, z) \in R^3 \,/\, 4x - 3y + z = 0\}$.

→ Let $W = \{(x, y, z) \in R^3 \,/\, 4x - 3y + z = 0\}$.

By def^n $(0, 0, 0) \in R^3$
$\therefore (0, 0, 0) \in W \subseteq R^3$
$\therefore$ W is non-empty subset of R^3. —— ①

① $u_1, u_2 \in W$, $u_1 = (x_1, y_1, z_1)$, $u_2 (x_2, y_2, z_2)$
$4x_1 - 3y_1 + z_1 = 0$, $4x_2 - 3y_2 + z_2 = 0$

$\therefore u_1 + u_2 = (x_1, y_1, z_1) + (x_2, y_2, z_2)$

$= (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$\therefore 4(x_1 + x_2) - 3(y_1 + y_2) + (z_1 + z_2)$
$= (4x_1 - 3y_1 + z_1) + (4x_2 - 3y_2 + z_2)$
$= 0 + 0$
$= 0$

$\therefore u_1 + u_2 \in W \quad \forall \quad u_1, u_2 \in W$

Let α be a scalar and $u = (x, y, z) \in W$
$\therefore 4x - 3y + z = 0$.

$\alpha u = \alpha (x, y, z) = (\alpha x, \alpha y, \alpha z)$

$\therefore 4(\alpha x) - 3(\alpha y) + (\alpha z) = \alpha (4x - 3y + z)$
$= \alpha (0)$
$= 0$.

$\therefore \alpha u \in W$
$\therefore$ W is subspace w.r.t. R^3

8) Show that intersection of any two subspaces is again a subspace

→ Let V(F) be a vector space.
Let W_1 and W_2 be two subspaces of V

To prove that $W_1 \cap W_2$ is a subspace of V

Since W_1 and W_2 are subspaces of V

$\therefore$ $0 \in W_1$ and $0 \in W_2$

$\therefore$ $0 \in W_1 \cap W_2 \subseteq W_1 \subseteq V$

$\therefore$ $W_1 \cap W_2$ is a ~~m~~ non-empty subset of V —— ①

Let $u, v \in W_1 \cap W_2$

$\Rightarrow$ $u, v \in W_1$ and $u, v \in W_2$

$\Rightarrow$ $u+v \in W_1$ and $u+v \in W_2$ $\because$ since W_1 & W_2 a[re] subspaces

$\Rightarrow$ $u+v \in W_1 + W_2$

$\therefore$ $u, v \in W_1 \cap W_2 \Rightarrow u+v \in W_1 \cap W_2$

$\therefore$ $W_1 \cap W_2$ is closed w.r.t. vector addition. —②

Let $\alpha \in F$ and $u \in W_1 \cap W_2$

$\Rightarrow$ $u \in W_1$, $u \in W_2$

$\Rightarrow$ $\alpha u \in W_1$, $\alpha u \in W_2$

$\Rightarrow$ $\alpha u \in W_1 \cap W_2$

$\therefore$ $\alpha u \in W_1 \cap W_2$ $\forall$ $u \in W_1 \cap W_2$ & all scalar α

$\therefore$ $W_1 \cap W_2$ is closed w.r.t. scalar multiplication —③

$\therefore$ from ①, ②, ③

$W_1 \cap W_2$ is a subspace of V

$\therefore$ Intersection of any two subspaces is again a subspace.

Note -

But union of two subspaces need not be a subspace.

Consider the vector space $V = R^3 = \{(x, y, z) / x, y, z \in R\}$

subspace of V are $U = \{(x,0,0) / x \in R\}$
$W = \{(0,y,0) / y \in R\}$

$\therefore U \cup W = \{(x,0,0) \text{ or } (0,y,0) / x, y \in R\}$

$= \{(x,y,0) / \text{either } x=0 \text{ or } y=0\}$

By def^n $U \cup W$ is non-empty subset of R^3
$(2,0,0), (0,1,0) \in U \cup W$

$\therefore (2,0,0) + (0,1,0) = (2,1,0) \notin U \cup W$
$\therefore U \cup W$ is not closed w.r.t. vector addition
$\therefore U \cup W$ is not sub-space of R^3.

2) $U \cup W$ is a subspace of vector space V iff either $U \subset W$ or $W \subset U$

9) $V = \{(z_1, z_2, z_3) / z_1, z_2, z_3 \in C\} = C^3$ be the vector space over C check whether
i) $W_1 = \{(z_1, z_2, z_3) / z_1 \text{ is real number}\}$

ii) $W_2 = \{(z_1, z_2, z_3) / z_1 + z_2 = 0\}$

iii) $W_3 = \{(z_1, z_2, z_3) / z_1 + z_2 = 1\}$ is a subspaces of V or not.

→ Consider, $V = \{(z_1, z_2, z_3) / z_1, z_2, z_3 \in C\}$

i) $W_1 = \{(z_1, z_2, z_3) / z_1 \text{ is real number}\}$.

For $(0,0,0) \in W_1 \subseteq \mathbb{C}^3$

$\therefore W_1$ is non-empty subset of $\mathbb{C}^3$ —— ①

Let $u = (z_1, z_2, z_3)$, $v = (z_1', z_2', z_3') \in W_1$

$\therefore u+v = (z_1, z_2, z_3) + (z_1', z_2', z_3')$
$= (z_1+z_1', z_2+z_2', z_3+z_3') \in W_1$

$\therefore u+v \in W_1 \quad \forall\, u, v \in W_1$

$\therefore W_1$ is closed w.r.t. vector addition.

Let $\alpha = 2+i \in \mathbb{C}$, $u = (2, 1+i, 3) \in W_1$

$\therefore \alpha u = (2+i)(2, 1+i, 3)$
$= (4+2i, 3i+1, 6+3i) \notin W_1$

$\therefore \alpha \in \mathbb{C}, u \in W_1 \Rightarrow \alpha u \notin W_1$

$\therefore W_1$ is not closed w.r.t. scalar multiplication.

$\therefore W_1$ is not subspace of $\mathbb{C}^3$.

ii) $W_2 = \{(z_1, z_2, z_3) / z_1 + z_2 = 0\}$
$W_2 = \{(z_1, -z_1, z_3) / z_1, z_3 \in \mathbb{C}\}$
For $(0,0,0) \in W_2 \subseteq \mathbb{C}^3$

$\therefore W_2$ is non-empty subset of $\mathbb{C}^3$ —— ①

Let $u = (z_1, -z_1, z_3)$, $v = (z_1', -z_1', z_3) \in W_2$

$\therefore u+v = (z_1, -z_1, z_3) + (z_1', -z_1', z_3)$
$= (z_1+z_1', -(z_1+z_1'), z_3+z_3') \in W_2$

$\therefore u+v \in W_2 \quad \forall \; u, v \in W_2$

$\therefore W_1$ is closed w.r.t. vector addition.

Let $\alpha \in C$ and $u \in W_2$

$\therefore \alpha u = \alpha (z_1, -z_1, z_3)$

$= (\alpha z_1, -\alpha z_1, \alpha z_3)$

$\therefore \alpha \in C, u \in W_2 \Rightarrow \alpha u \in W_2$

$\therefore W_2$ is closed w.r.t. scalar multiplication. — ③

$\therefore$ from ①, ②, ③

W_2 is a subspace of C^3.

iii) $W_3 = \{(z_1, z_2, z_3) / z_1 + z_2 = 1\}$

$\therefore W_3 = \{(z_1, z_2, z_3) / z_2 = 1 - z_1\}$

$\therefore W_3 = \{(z_1, 1-z_1, z_3) / z_1, z_3 \in C\}$

$\therefore$ For $(0,0,0) \notin W_3 \subseteq C^3$

~~$\therefore W_3$ is non-empty subset of~~ C^3 — ①

~~Let $u = (z_1, 1-z_1, z_3)$~~

$\therefore W_3$ is not a subspace of C^3.

* SUM of two subspaces

W_1 and W_2 are two sub-spaces of $V(F)$ then we define $W_1+W_2 = \{\omega_1+\omega_2 / \omega_1 \in W_1, \omega_2 \in W_2$

* Show that sum of two subspaces of a vector space is a subspace of vector space.

→ Let $V(F)$ is a vector space

Let W_1, W_2 be two subspaces of V. ~~by~~

By defn,

$W_1+W_2 = \{\omega_1+\omega_2 / \omega_1 \in W_1, \omega_2 \in W_2\}$.

W_1+W_2 is non-empty subset of V —— ①.

Let $x, y \in W_1+W_2$

$x = u_1+u_2, \; y = \omega_1+\omega_2$

where, $u_1, \omega_1 \in W_1, \; u_2, \omega_2 \in W_2$

$\therefore x+y = (u_1+u_2)+(\omega_1+\omega_2)$

$= (u_1+\omega_1)+(u_2+\omega_2) \in W_1+W_2$

$\because \omega_1$ & ω_2 are subspaces.

$\therefore W_1+W_2$ is closed w.r.t. vector addition. —②

Let $\alpha \in F$, $x = (u_1+u_2) \in W_1+W_2$ where $u_1 \in W_1$, $u_2 \in W_2$

$\alpha x = \alpha(u_1+u_2)$

$= \alpha u_1 + \alpha u_2 \in W_1+W_2$

$\because \omega_1$ & ω_2 are subspaces of V.

$\therefore \alpha x \in W_1+W_2 \;\; \forall \alpha \in F, x \in W_1+W_2$.

$\therefore W_1+W_2$ is closed w.r.t. scalar multiplication —③

from ①, ②, ③:

W_1+W_2 is subspace of V

$\therefore$ Sum of two subspaces of vector space is a subspace of vector space.

* Direct Sum –

We say that a vector space V is the direct sum of two subspaces W_1 and W_2 if i) $V = W_1 + W_2$

ii) Every $v \in V$ can be expressed uniquely as the sum $\omega_1 + \omega_2$ where $\omega_1 \in W_1$ and $\omega_2 \in W_2$

In that case we write

$$V = W_1 \oplus W_2$$

* Theorem –

$V = W_1 \oplus W_2$ iff $V = W_1 + W_2$, $W_1 \cap W_2 = \{0\}$

$\rightarrow$ Let V be a vector space. W_1, W_2 be two subspaces of V

First suppose that $V = W_1 \oplus W_2$

By defn of direct sum

i) $V = W_1 + W_2$

ii) Every element ~~ea~~ in V can be expressed uniquely as the sum $\omega_1 + \omega_2$ where $\omega_1 \in W_1$, $\omega_2 \in W_2$.

$\therefore$ Only we have to prove that $W_1 + W_2 = \{0\}$.

Let if possible $0 \neq x \in W_1 \cap W_2$

$\Rightarrow x \in W_1, \quad x \in W_2$

$\Rightarrow x = x + 0, \quad x \in W_1, \quad 0 \in W_2$

$\Rightarrow x = 0 + x, \quad 0 \in W_1, \quad x \in W_2$

Thus $x \in V$ can be expressed in atleast two different ways as sum of an element of W_1 and an element of W_2.

This contradicts to the fact that V is the direct sum of W_1 and W_2.

Hence 0 is the only vector common to both W_1 and W_2.

$\therefore W_1 \cap W_2 = \{0\}$

Conversly suppose that $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$.

To prove that $V = W_1 \oplus W_2$.

since $V = W_1 + W_2 \Rightarrow$ implies that each element of V can be expressed as sum of an element of W_1 and an element of W_2.

Now to show that this expression is unique

Let if possible $v \in V$

$\therefore v = \omega_1 + \omega_2,\ v = \omega_1' + \omega_2'$

$\therefore \omega_1 + \omega_2 = \omega_1' + \omega_2'$

$\therefore \omega_1 - \omega_1' = \omega_2' - \omega_2$

since $\omega_1, \omega_1' \in W_1 \Rightarrow \omega_1 - \omega_1' \in W_1$

and $\omega_2, \omega_2' \in W_2 \Rightarrow \omega_2 - \omega_2' \in W_2$

since W_1 & W_2 are subspaces.

but $W_1 \cap W_2 = \{0\}$.

$\therefore \omega_1 - \omega_1' = 0 = \omega_2' - \omega_2$

$\Rightarrow \omega_1 = \omega_1'$ & $\omega_2 = \omega_2'$

$\therefore$ Every element in V can be uniquely expressed as the sum $\omega_1 + \omega_2$, $\omega_1 \in W_1$, $\omega_2 \in W_2$

$\therefore V = W_1 \oplus W_2$.

* Quotient Space –

Def^n – If W is a subspace of a vector space V(F). Let $\frac{V}{W}$ be the set of all cosets is called the quotient space over F i.e. quotient space $\frac{V}{W} = \{W + v \mid v \in V\}$ is a vector space over F. w.r.t. addition & scalar multiplication defined by

$$(W + x) + (W + y) = W + (x + y)$$
$$\alpha(W + x) = W + \alpha x \quad \forall \; x, y \in V, \alpha \in F.$$

Note –
$W + 0 = W$ is an additive identity in $\frac{V}{W}$ and
$W + (-x)$ is the inverse of $W + x$.

- Homomorphism or Linear Transformation:

Let V and U be two vector spaces over the same field F then the mapping $T: V \to U$ is called a Homomorphism or a linear transformation if $T(x+y) = T(x) + T(y)$ and
$T(\alpha x) = \alpha T(x) \quad \forall \; x, y \in V, \alpha \in F$

or $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \quad \forall \; x, y \in V, \alpha, \beta \in F$

Note – The linear transformation T is said to be isomorphism if it is one to one and onto.

Show that the transformation $T : R^2 \to R^2$ defined by $T(x_1, x_2) = (x_2, x_1)$ is a linear Transformation

→ Given, $T : R^2 \to R^2$ defined by $T(x_1, x_2) = (x_2, x_1)$

1) Let $u = (x_1, x_2)$, $v = (y_1, y_2) \in R^2$

$$T(u+v) = T((x_1, x_2) + (y_1, y_2))$$

$$= T(x_1 + y_1, x_2 + y_2)$$

$$= (x_2 + y_2, x_1 + y_1)$$

$$= (x_2, x_1) + (y_2, y_1)$$

$$= T(x_1, x_2) + T(y_1, y_2)$$

$$= T(u) + T(v) \quad \forall \; u, v \in R^2$$

Let $u = (x_1, x_2)$ and α be a scalar $\therefore \alpha \in R$

$$T(\alpha u) = T(\alpha(x_1, x_2))$$

$$= T(\alpha x_1, \alpha x_2)$$

$$= (\alpha x_2, \alpha x_1)$$

$$= \alpha(x_2, x_1)$$

$$= \alpha T(x_1, x_2)$$

$$T(\alpha u) = \alpha T(u) \quad \forall u \in R^2 \;\; \therefore \alpha \in R$$

$\therefore$ From ① & ② T is alinear transformation.

2) Show that $T: R^2 \to R^3$ defined by $T(x_1, x_2) = (x_1, x_1+x_2, x_2)$ is a linear transformation.

→ Given $T: R^2 \to R^3$ is defined by

$$T(x_1, x_2) = (x_1, x_1+x_2, x_2)$$

Let $u = (x_1, x_2)$, $v = (y_1, y_2)$

$$T(u+v) = T(x_1+y_1, x_2+y_2)$$

$$= (x_1+y_1, x_1+y_1+x_2+y_2, x_2+y_2)$$

$$= (x_1, x_1+x_2, x_2) + (y_1, y_1+y_2, y_2)$$

$$= T(x_1, x_2) + T(y_1, y_2)$$

$$= T(u) + T(v) \quad \forall u, v \in R^2$$

Let $u = (x_1, x_2)$ and α be a scalar $\alpha \in R$

$$T(\alpha u) = T(\alpha(x_1, x_2))$$

$$= T(\alpha x_1, \alpha x_2)$$

$$= \alpha(x_1, x_1+x_2, x_2)$$

$$= \alpha T(x_1, x_2)$$

$$= \alpha T(u) \quad u \in R^2$$

$\therefore$ T is a linear transformation.

3) Let V be the set of all differentiable functions of x define $D : V \rightarrow V$ by $D(f) = \frac{df}{dx}$, $\forall f \in V$

→ Given ① $D : V \rightarrow V$ defined by

② $D(f) = \frac{df}{dx}$ $\forall f \in V$

Let, $f, g \in V$

$\therefore D(f+g) = \frac{d}{dx}(f+g)$

$= \frac{df}{dx} + \frac{dg}{dx}$

$= D(f) + D(g)$ — ①

Let $f \in V$, $\alpha \in F$

$\therefore D(\alpha f) = \frac{d}{dx}(\alpha f) = \alpha \frac{df}{dx} = \alpha D(f)$ — ②

$\therefore$ From ①, ②

$D\ \cancel{(f)}$ is a linear transformation.

4) $\int (\alpha f + \beta y) = \alpha \int f + \beta \int y$

$\therefore \int$ operator is a L.T.

$T(u)+T(v) = (1+x_1, x_2) + (1+x_1$
$=$

5) Identity map $I : V \to V$ defined by $I(u) = u$, $\forall u \in$ is a L.T.

→ $I(u+v) = u+v$
$= I(u) + I(v)$ —— ①
$I(\alpha u) = \alpha u = \alpha I(u)$ —— ②

6) Zero function, $O : V \to V$ defined by $O(u) = 0$ $\forall u \in$ is L.T.

→ $O(u+v) = 0$
$= 0 + 0$
$= O(u) + O(v)$

$O(\alpha u) = 0$
$= \alpha \cdot 0$
$= \alpha\, O(u)$

7) Check whether the following transformations is linear or not

$T : R^2 \to R^2$ such that $T(x_1, x_2) = (1+x_1, x_2)$

→ Let $u = (x_1, x_2)$, $v = (y_1, y_2)$

$T(u+v) = T(x_1+y_1, x_2+y_2)$

$= (1+x_1+y_1, x_2+y_2)$

$T(u) + T(v) = T(x_1, x_2) + T(y_1, y_2)$

$= (1+x_1, x_2) + (1+y_1, y_2)$

$= (2 + x_1 + y_1, x_2 + y_2)$

$\therefore T(u+v) \neq T(u) + T(v)$

$\therefore$ T is not a L.T.

ii) $T : R \to R^3$ such that $T(x) = (x, x^2, x^3)$

Let $x, y \in R$

$T(x+y) = \cancel{T}[(x+y), (x+y)^2, (x+y)^3]$

$T(x) + T(y) = (x, x^2, x^3) + (y + y^2, y^3)$

$= (x+y, x^2+y^2, x^3+y^3)$

$\therefore T(x+y) \neq T(x) + T(y)$

$\therefore$ T is not a L.T.

iii) $T : C \to C$ such that $T(z) = \bar{z}$ i.e. $T(x+iy) = x-iy$

iv) $T : R^2 \to R^2$ such that, $T(x, y) = (x, -y)$

$T : C \to C$ $T(x+iy) = x$

$T : R^3 \to R^4$ such that $T(x_1, x_2, x_3) = (x_1, x_1+x_2, x_1+x_2+x_3, x_2$

iii) Given, $T: C \to C$ such that, $T(z) = \overline{z}$

i.e. $T(x+iy) = x - iy$

1) Let $u = a+ib$, $v = a'+ib'$

$T(u+v) = T(a+ib+a'+ib')$

$= T(a+a'+i(b+b'))$

$= (a+a'-i(b+b'))$

$T(u)+T(v) = T(a+ib) + T(a'+ib')$

$= (a-ib) + (a'-ib')$

$= a+a' - i(b+b')$

$T(u+v) = T(u)+T(v)$

2) Let $u = a+ib$, α be the scalar

$T(\alpha u) = T(\alpha(a+ib))$

$= T(\alpha a + \alpha ib)$

$= (\alpha a - \alpha ib)$

$= \alpha(a-ib)$

$= \alpha T(a+ib)$

$= \alpha T(u)$

From ① & ② T is L.T.

v) Given $T: C \to C$ s.t. $T(x+iy) = x$

1) Let $u = a+ib$, $v = a'+ib' \in C$

$T(u+v) = T(a+ib, a'+ib')$

$= T(a+a'+i(b+b'))$

$= a+a'$

$T(u)+T(v) = T(a+ib) + T(a'+ib')$

$= (a) + a'$

$\therefore T(u+v) = T(u)+T(v)$

2) Let $u = a+ib \in C$ & α is scalar

$T(\alpha u) = T(\alpha(a+ib))$

$= T(\alpha a + \alpha ib)$

$= \alpha a$
$= \alpha T(a+ib)$
$= \alpha T(u)$
From ① & ② T is L.T.

vi) Given $T: R^3 \to R^4$ such that,
$T(x_1, x_2, x_3) = (x_1, x_1+x_2, x_1+x_2+x_3, x_3)$
1) Let $u = (x_1, x_2, x_3)$ & $v = (y_1, y_2, y_3) \in R^3$
$T(u+v) = T((x_1, x_2, x_3) + (y_1, y_2, y_3))$
$= T(x_1+y_1, x_2+y_2, x_3+y_3)$
$= (x_1+y_1, x_1+y_1+x_2+y_2, x_1+y_1+x_2+y_2+x_3+y_3, x_3+y_3)$
$= (x_1+y_1, x_1+x_2+y_1+y_2, x_1+x_2+x_3+y_1+y_2+y_3, x_3+y_3)$

Now,
$T(u)+T(v) = T(x_1, x_2, x_3) + T(y_1, y_2, y_3)$
$= (x_1, x_1+x_2, x_1+x_2+x_3, x_3) + (y_1, y_1+y_2, y_1+y_2+y_3, y_3)$
$= (x_1+y_1, x_1+x_2+y_1+y_2, x_1+x_2+x_3+y_1+y_2+y_3, x_3+y_3)$

$\therefore T(u+v) = T(u) + T(v)$

2) Let $u = (x_1, x_2, x_3)$ α be scalar,
$T(\alpha u) = T(\alpha(x_1, x_2, x_3))$
$= T(\alpha x_1, \alpha x_2, \alpha x_3)$
$= (\alpha x_1, \alpha x_1+\alpha x_2, \alpha x_1+\alpha x_2+\alpha x_3, \alpha x_3)$
$= \alpha T(x_1, x_2, x_3)$
$= \alpha T(u)$

From ① & ②
T is L.T.

* Theorem

Let $T: V \to U$ be a L.T. then i) $T(0) = 0$.

ii) $T(-x) = -T(x)$, iii) $T(x-y) = T(x) - T(y)$

iii) $T(x-y) = T(x) - T(y)$

→ i)

$$0 + 0 = 0$$

$$T(0+0) = T(0)$$

$$T(0) + T(0) = T(0) \quad T \text{ is L.T.}$$

$$T(0) + T(0) = T(0) + 0. \quad 0 \text{ is an identity}$$

$$T(0) = 0 \quad \therefore \text{ by cancellation law}$$

ii) $$x + (-x) = 0$$

$$T(x + (-x)) = T(0)$$

$$T(x) + T(-x) = 0 \quad \because T \text{ is L.T. } \& \ T(0) = 0$$

$$\text{lly} \quad T(-x) + T(x) = 0$$

$$\therefore \cancel{T(x} \ T(-x) = -T(x) \quad \forall \ x \in V.$$

iii) Let $x, y \in V$

$$\therefore T(x-y) = T(x + (-y))$$

$$= T(x) + T(-y) \quad \because T \text{ is L.T.}$$

$$= T(x) - T(y) \quad \because \text{from } 2. \ \forall x, y \in V$$

* Null space of T or KerT

Def^n - Let $T: V \to U$ be Linear Transformation (or Homomorphism) then Kernel of T is defined by,

$$\text{Ker}T \ \{ v \in V / T(v) = 0 \}.$$

It is also called null space of T & $\text{Ker}T \subseteq V$

* Theorem

Let $T: V \to U$ be a linear transformation (Homomorphism) then $\ker T$ is a subspace of V

→ Let $T: V \to U$ be a linear transformation.
By defⁿ $\ker T = \{v \in V / T(v) = 0\}$

since $T(0) = 0$
$\therefore \; 0 \in \ker T \subseteq V$

$\therefore$ $\ker T$ is non-empty subset of V —— ①

Let α, β be a scalars and let $u, v \in \ker T$
$\therefore \; T(u) = 0$ and $T(v) = 0$.

consider $T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$ $\because T$ is L.T
$= \alpha(0) + \beta(0)$
$= 0 + 0$
$= 0$

$\therefore (\alpha(u) + \beta v) \in \ker T$ $\forall$ $u, v \in \ker T$ & for all scalars α, β —②

$\therefore$ From ① & ②.
$\ker T$ is subspace of V

* Theorem -

Let $T: V \to U$ be a linear transformation (Homomor.
then $\ker T = \{0\}$ iff T is one to one.

→ Let $T: V \to U$
and by defⁿ $\ker T = \{v \in V \mid T(v) = 0\}$

First suppose that,
$\ker T = \{0\}$ to prove that T is one to one.

Let $u, v \in V$ such that,
$\Rightarrow T(u) = T(v)$
$\Rightarrow T(u) - T(v) = 0$.
$\Rightarrow T(u-v) = 0$ $\because$ T is L.T.
$\Rightarrow u - v \in \ker T = \{0\}$
$\Rightarrow u - v = 0$
$\Rightarrow u = v$ $\forall u, v \in V$
$\therefore$ T is one to one.

Conversly suppose that, T is one to one.
To prove that $\ker T = \{0\}$.

Let $u \in \ker T$
$\Rightarrow T(u) = 0$
$T(u) = T(0)$ $T(0) = 0$.
$\cancel{\Rightarrow}$ $u = 0$ T is one to one

$\therefore \ker T = \{0\}$.

Example 1) Zero transformation $0 : V \to V$ defined by
$0(v) = 0$ $\forall v \in V$

$\rightarrow$ $\ker 0 = V$ = Domain.

2) Identity function $I : V \to V$ defined ~~by~~, by,
$I(v) = v$ $\forall v \in V$.

$\rightarrow$ $\ker I = \{0\}$.

3) Differential operator $D(f) = \frac{df}{dx}$

→ Since $D(f) = 0$ iff f is a constant function

∴ Ker D is the set of all constant functions.

4) For Integral operator $\int$

∴ $\ker \int = \{0\}$

5) Find KerT if the L.T. $T: R^2 \to R^3$ defined by $T(x, y) = (x, x+y, y)$.

→ Given, $T: R^2 \to R^3$ defined by,

$T(x, y) = (x, x+y, y)$

By def^n $\ker T = \{v \in V / k = T(v) = 0\}$ ∴

$$\ker T = \{u \in R^2 / T(u) = 0\}$$
$$= \{(x, y) \in R^2 / T(x, y) = 0\}$$
$$= \{(x, y) \in R^2 \mid T(x, y) = (0, 0, 0)\}$$
$$= \{(x, y) \in R^2 / (x, x+y, y) = (0, 0, 0)\}$$
$$= \{(x, y) \in R^2 / x = 0, y = 0\}$$

$\ker T = (0, 0) \in R^2$

$\ker T = \{(0, 0)\}$.

* Range of T

Let $T: V \to U$ be a linear transformation then range of T is defined to be $T(V) = \{T(v)\}_{v \in V}$

i.e. $T(V) = \{u \in U / u = T(v), v \in V\}$

$= R_T$

and $R_T \subseteq U$

* Theorem -

Let $T: V \to U$ be a L.T. then range of T is a subspace of U

→ Let $T: V \to U$ be a linear transformation.

By def^n $R_T = T(V) = \{T(v) / v \in V\}$

$T(V) = \{u \in U / u = T(v), v \in V\}$.

since $T(0) = 0$

$\therefore T(0) \in R_T \subseteq U$

$\therefore$ R_T is non-empty subset of U —— ①

Let α, β be two scalars and let $T(x), T(y) \in R_T$ where $x, y \in V$

$\therefore \alpha \cdot T(x) + \beta \cdot T(y) = T(\alpha x) + T(\beta y)$

$= T(\alpha x + \beta y) \in R_T$

$\therefore$ $\alpha T(x) + \beta T(y) \in R_T$ for all scalars α, β

& $\forall$ $T(x), T(y) \in R_T$ —— ②

$\therefore$ From ① and ② R_T is subspace of U.

* If $T: V \to U$ is a L.T. then - i) $\ker T = \{0\}$ iff T is one - to one ii) $R_T = T(V) = U$ iff T is onto.

Note:

- Let $T: V \to U$ be a L.T. then $\frac{V}{\ker T} \cong R_T = T(V)$

This is called fundamental theorem of homomorphism for vector spaces.

- If A and B are two subspaces of vector space $V(F)$ then $\frac{A+B}{A} \cong \frac{B}{A \cap B}$

- If $A+B$ is the direct sum then $A \cap B = (0)$

If $\frac{A}{(0)} \cong \frac{A \oplus B}{B}$ or $\frac{B}{(0)} \cong \frac{A \oplus B}{A}$

But $\frac{A}{(0)} = \{(0) + x \mid x \in A\}$

$= \{x / x \in A\}$

$= A$

$A \cong \frac{A \oplus B}{B}$ or $B \cong \frac{A \oplus B}{A}$

- $V(F)$ is vector space then $\{0\}$ and V are subspaces of V.

2. - Natural Homomorphism or Quotient map

Let W be a subspace of V then their exists an onto linear transformation $\theta: V \to \frac{V}{W}$ such that

$\ker \theta = W$

Note:

- Let $T: V \to U$ be a L.T. then $\frac{V}{\text{Ker}T} \cong R_T = T(V)$

This is called fundamental theorem of homomorphism for vector spaces.

- If A and B are two subspaces of vector space $V(F)$ then $\frac{A+B}{A} \cong \frac{B}{A \cap B}$

- If $A+B$ is the direct sum then $A \cap B = (0)$

If $\frac{A}{(0)} \cong \frac{A \oplus B}{B}$ or $\frac{B}{(0)} \cong \frac{A \oplus B}{A}$

But $\frac{A}{(0)} = \{ (0) + x \mid x \in A \}$

$= \{ x / x \in A \}$

$= A.$

$A \cong \frac{A \oplus B}{B}$ or $B \cong \frac{A \oplus B}{A}$

- $V(F)$ is vector space then $\{0\}$ and V are subspaces of V.

2. • Natural Homomorphism or Quotient map

Let W be a subspace of V then their exists an onto linear transformation $\theta: V \to \frac{V}{W}$ such that

$\text{Ker}\,\theta = W$

→ Let V be a vector space and W be a subspace of V

$\therefore \frac{V}{W}$ is the quotient space defined $\theta: V \to \frac{V}{W}$

by $\theta(x) = W + x \quad \forall x \in V$.

i) θ is well defined.

Let $x, y \in V$ such that

$$x = y$$

$$W + x = W + y \qquad \because W \text{ is a subspace.}$$

$$\theta(x) = \theta(y)$$

$\therefore \theta$ is well - defined.

ii) θ is L.T.

Let $x, y \in V$ and α, β be scalars.

$\therefore \theta(\alpha x + \beta y) = W + (\alpha x + \beta y)$ .. By defn

$= (W + \alpha x) + (W + \beta y)$

$(\because (W + x) + (W + y) = W + (x+y))$

$= \alpha(W + x) + \beta(W + y)$

$= \alpha\, \theta(x) + \beta\theta(y) \quad \forall x, y \in V$

$\therefore \theta$ is a L.T.

iii) θ is onto

For $W + x \in \frac{V}{W}$ there exist $x \in V$ such that

$\theta(x) = W + x \quad \forall x \in V$.

$\therefore \theta$ is onto.

Now we have to prove that $\ker\theta = W$.

Let $x \in \ker\theta$ iff $\theta(x) = W$

$\Leftrightarrow W + x = W$

$\Leftrightarrow x \in W$

$\therefore \ker\theta = W$

This L.T. is called natural homomorphism or the quotient map.

* Linear span or span :-

Def^n - Let $V(F)$ be a vector space. $v_i \in V$, $\alpha_i \in F$ then elements of the type

$$\sum_{i=1}^{n} \alpha_i v_i = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

are called linear combination of $v_1, v_2, v_3, \ldots v_n$ over F.

Let S be a non-empty subset of V then the set of all linear combinations of finite number of elements of S is called linear span of S. It is denoted by $L(S)$ or $[S]$ or $\langle S \rangle$

i.e. $L(S) = \left\{ \sum_{i=1}^{n} \alpha_i v_i \;/\; v_i \in S, \alpha_i \in F \right\}$

examples

Find $L(S)$ if $S = \{(1,0), (0,2)\} \subset R^2$

$\rightarrow$ R^2 is vector space over R

Let $S = \{(1,0), (0,2)\}$

By def^n $L(S) = \{\alpha(1,0) + \beta(0,2) \;/\; \alpha, \beta \in R\}$

$$L(S) = \{(\alpha, 0) + (0, 2\beta) / \alpha, \beta \in R\}$$

$$= \{(\alpha, 2\beta) / \alpha, \beta \in R\}$$

$$= \{(x, y) / x = \alpha, y = 2\beta \in R\}$$

$$L(S) = R^2$$

2) In R^3 find $\langle (1,0,0), (0,1,0) \rangle$. Check whether i) $(2,3,0) \in \langle (1,0,0), (0,1,0) \rangle$ or not

→ R^3 is a vector space.

Let $S = \{(1,0,0), (0,1,0)\}$.

$$\therefore L(S) = \{\alpha(1,0,0) + \beta(0,1,0) / \alpha, \beta \in R\}$$

$$= \{(\alpha, 0, 0) + (0, \beta, 0) / \alpha, \beta \in R\}$$

$$= \{(\alpha, \beta, 0) / \alpha, \beta \in R\}$$

$$= \{(x, y, 0) / x = \alpha, y = \beta \in R\}$$

$= XY$ plane.

Suppose $(2,3,0) \in L(S)$

$\therefore$ $(2,3,0)$ is a linear combination of $(1,0,0), (0,1,0)$

Let α, β be scalars such that

$\therefore (2,3,0) \in \langle S \rangle$.

$\therefore \; (2,3,0) = \alpha(1,0,0) + \beta(0,1,0)$
$= \alpha(\alpha,0,0) + (0,\beta,0)$
$= (\alpha,\beta,0)$

$\alpha = 2 \;,\; \beta = 3$

$(2,3,0) = 2(1,0,0) + 3(0,1,0)$
$\therefore (2,3,0) \in \langle S \rangle$

ii) $(2,3,1) \in \langle (1,0,0),(0,1,0) \rangle$ or not.
suppose $(2,3,1) \in \langle S \rangle$
$\therefore (2,3,1)$ ~~is a~~ $= \alpha(1,0,0) + \beta(0,1,0)$
$= (\alpha,0,0) + (0,\beta,0)$
$(2,3,1) = (\alpha,\beta,0)$

$\Rightarrow \alpha = 2, \beta = 3, 0 = 1$ which is not possible.
$\therefore (2,3,1) \notin \langle S \rangle$
$\therefore$ Our supposition is wrong
$\therefore (2,3,1) \notin (S)$

3) Let $V = R^4$ Let $S = \{(2,0,01),(-1,0,1,0)\}$. Find $L(S)$ and check whether $(1,2,3,4) \in L(S)$ or not.

→ R^4 is a vector space.
Let $S = \{(2,0,0,1),(-1,0,1,0)\}$

$L(S) = \{\alpha(2,0,0,1) + \beta(-1,0,1,0) \,/\, \alpha,\beta \in R\}$

$= \{(2\alpha,0,0,\alpha) + (-\beta,0,\beta,0) \,/\, \alpha,\beta \in R\}$

$= \{(2\alpha-\beta,0,\beta,\alpha) \,/\, \alpha,\beta \in R\}$

suppose $(1,2,3,4) \in L(S)$

$\therefore (1,2,3,4) = \alpha(2,0,0,1) + \beta(-1,0,1,0)$

$(1,2,3,4) = (2\alpha-\beta, 0, \beta, \alpha)$

$\Rightarrow 2\alpha - \beta = 1, \; 2 = 0, \; \beta = 3, \; 4 = \alpha$

which is not possible.

$\therefore$ our supposition is wrong

$(1,2,3,4) \notin L(S)$.

ii) $(0,0,2,1) \in L(S)$ or not;

suppose $(0,0,2,1) \in L(S)$

$\therefore (0,0,2,1) = \alpha(2,0,0,1) + \beta(-1,0,1,0)$

$(0,0,2,1) = (2\alpha-\beta, 0, \beta, \alpha)$

$\Rightarrow 0 = 2\alpha - \beta, \; 0 = 0, \; 2 = \beta, \; 1 = \alpha$

$\therefore (0,0,2,1) \in L(S)$.

4) Let $S = \{(1,4), (0,3)\}$ be subset of R^2 over R show that $(2,3) \in L(S)$

* Theorem

L(S) is the smallest subspace V containing S.

→ Let V(F) be a vector space.

Let S be a non-empty subset of V.

By def^n,

$$L(S) = \left\{ \sum_{i=1}^{n} \alpha_i v_i \;/\; v_i \in S,\ \alpha_i \in F \right\}$$

By def^n of L(S), L(S) is non-empty subset of — ①

Let $x, y \in L(S)$ and $\alpha, \beta \in S$.

$$\therefore x = \sum_{i=1}^{n} \alpha_i v_i = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

$$y = \sum_{j=1}^{m} \beta_j v_j' = \beta_1 v_1' + \beta_2 v_2' + \cdots + \beta_m v_m'$$

where $\alpha_i, \beta_j \in F \quad \forall \; i, j \in N$

and $v_i, v_j' \in S$

$\therefore \alpha x + \beta y = \alpha(\alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_n v_n)$
$+ \beta(\beta_1 v_1' + \beta_2 v_2' + \cdots \beta_m v_m')$

$= (\alpha\alpha_1)v_1 + (\alpha\alpha_2)v_2 + \cdots + (\alpha\alpha_n)v_n$
$+ (\beta\beta_1)v_1' + (\beta\beta_2)v_2' + \cdots (\beta\beta_m)v_m'$

$\therefore \alpha x + \beta y$ is a linear combination of finite number of elements in $L(S)$

$\therefore \alpha x + \beta y \in L(S) \ \forall \ \alpha, \beta \in F$
and $\forall \ x, y \in L(S)$ —— ②

$\therefore$ From ① & ②
$L(S)$ is subspace of V.

For $v_1 \in S$
$v_1 = 1 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_n = 1 \cdot v_1 \quad , v_i \in S$

$\therefore v_1 \in L(S)$

$\therefore S \subseteq L(S) \quad \therefore L(S)$ is subspace of v containing S

Let W be any subspace of V containing S

Let $x \in L(S)$
$\therefore$ x is a linear combination of finite number of elements in S.

$\therefore x = \sum_{i=1}^{n} \alpha_i v_i \quad , \alpha_i \in f \quad , v_i \in S$

$= \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$

since $v_i \in S \subseteq W$

$\Rightarrow v_i \in W. \Rightarrow \alpha_i v_i \in W, \quad \forall i$ ∵ W is subspace

$\Rightarrow \sum_{i=1}^{n} \alpha_i v_i \in W$ ∵ W is subspace.

$\Rightarrow x \in W$

∴ $L(S) \subseteq W$

∴ $L(S)$ is smallest subspace of V containing S

* Theorem

If W is subspace of V then $L(W) = W$ and conversly

→ Let $V(F)$ be a vector space.
First suppose that W is subspace of V
We know that "W is non-empty subset of V then $L(W)$ is smallest subspace of V containing W."

∴ $W \subseteq L(W)$ ——— ①

Let $x \in L(W)$

$x = \sum_{i=1}^{n} \alpha_i v_i, \quad \alpha_i \in F, v_i \in W.$

$\Rightarrow x = \sum_{i=1}^{n} \alpha_i v_i \in W$ ∵ W is a subspace.

∴ $L(W) \subseteq W$ ——— ②

∴ From ① & ② $L(W) = W$.

conversly suppose $L(W) = W$.
since $L(W)$ is subspace of V
∴ W is subspace of V.

Note -

1) If S_1 and S_2 are subsets of V then

i) $S_1 \subseteq S_2 \Rightarrow L(S_1) \subseteq L(S_2)$

ii) $\cancel{L(S_1) \cup L(S}$ $L(S_1 \cup S_2) = L(S_1) + L(S_2)$

iii) $L\{S_1\} = L\{S_1\}$

iv) $L(L(L(S_1))) = L(S_1)$

2) If S_1 and S_2 are subspaces of V then $L(S_1 \cup S_2) = L(S_1) + L(S_2) = S_1 + S_2$

i.e. if S_1 and S_2 are subspaces of V then $S_1 + S_2$ is the subspace of V spanned by $S_1 \cup S_2$

* Finite Dimensional Vector Space [F.D. V.S.]

The vector space V is said to be finite dimensional over F if there exist a finit subset S of V such that, $V = L(S)$

eg. 1) Show that $R^2(R)$ is a finite dimention vector space

→ Let $R^2 = \{(x, y) / x, y \in R\}$

Let $S = \{(1,0), (0,1)\} \subseteq R^2$.

$L(S) = \{\alpha(1,0) + \beta(0,1) / \alpha, \beta \in R\}$

$= \{(\alpha, 0) + (0, \beta) / \alpha, \beta \in R\}$

$= \{(\alpha, \beta) / \alpha, \beta \in R\}$

$= R^2$.

$\therefore L(S) = R^2$

$\therefore$ Dimension of $R^2 = 2$.

$\therefore R^2(R)$ is finite Dimensional Vector space.

2) Show that $R^3(R)$ is a F.D.V.S.

$\rightarrow$

$R^3 = \{(x, y, z) / x, y, z \in R\}$

$S = \{(1,0,0), (0,1,0), (0,0,1)\} \subseteq R^3$

$L(S) = \{\alpha(1,0,0) + \beta(0,1,0) + \gamma(0,0,1) / \alpha, \beta, \gamma \in R\}$

$= \{(\alpha,0,0) + (0,\beta,0) + (0,0,\gamma) / \alpha, \beta, \gamma \in R\}$

$= \{(\alpha, \beta, \gamma) / \alpha, \beta, \gamma \in R\}$

$= R^3$

$L(S) = R^3$

$\therefore$ Dimension of $R^3 = 3$

$\therefore R^3(R)$ is finite Dimensional Vector space.

* Linear Dependence And Independence [L.D. & L.I.]

Def^n - Let $V(F)$ be a vector space. vectors v_1, v_2, ... v_n in V are said to be linearly dependent if there exist scalars $\alpha_1, \alpha_2, \ldots \alpha_n$ in F not all zero such that $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$

If $v_1, v_2, \dots v_n$ are not linearly dependent then these vectors are called linearly independent. i.e. $v_1, v_2 \dots v_n$ are linearly dependent if $\sum_{i=1}^{n} \alpha_i v_i = 0$ implies atleast one $\alpha_i \neq 0$.

And $v_1, v_2, \dots v_n$ are linearly independent if $\sum_{i=1}^{n} \alpha_i v_i = 0 \Rightarrow$ all $\alpha_i = 0$

A finite set $X = \{x_1, x_2, \dots x_n\}$ is said to be linearly dependent or linearly independent According as it's n members are linearly dependend or lineary independent.

eg. 1) Show that the vectors $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ are linearly independent in R^3.

→ Given vectors are $(1,0,0)$, $(0,1,0)$, $(0,0,1)$

Let α, β, γ be scalars such that,

$$\alpha(1,0,0) + \beta(0,1,0) + \gamma(0,0,1) = 0$$
$$\Rightarrow (\alpha,0,0) + (0,\beta,0) + (0,0,\gamma) = 0.$$
$$\Rightarrow (\alpha,\beta,\gamma) = (0,0,0)$$
$$\Rightarrow \alpha = 0 \; , \; \beta = 0 \; , \; \gamma = 0$$

$\therefore$ Given vectors are L.I. in R^3

2) Show that $S = \{(1,0,0), (0,1,0), (0,0,1), (2,3,4)\}$ is L.D. in R^3.

→ Given vectors are

$S = \{(1,0,0), (0,1,0), (0,0,1), (2,3,4)\}$.

Let $\alpha, \beta, \gamma, \delta$ be scalars such that,

$\alpha(1,0,0) + \beta(0,1,0) + \gamma(0,0,1) + \delta(2,3,4) = 0$

$\Rightarrow (\alpha,0,0) + (0,\beta,0) + (0,0,\gamma) + (2\delta, 3\delta, 4\delta) = 0$

$\Rightarrow (\alpha+2\delta, \beta+3\delta, \gamma+4\delta) = (0,0,0)$

$\Rightarrow \alpha+2\delta = 0, \quad \beta+3\delta=0, \quad \gamma+4\delta = 0.$

$\Rightarrow \alpha = -2\delta, \quad \beta = -3\delta, \quad \gamma = -4\delta.$

$\therefore$ Not all scalars $\alpha, \beta, \gamma, \delta$ are zero.

$\therefore$ S is linearly dependent.

3) Show that $S = \{(1,2,3), (0,1,2), (1,5,0), (1,0,0)\}$

—

Given vectors are

$S = \{(1,2,3), (0,1,2), (1,5,0), (1,0,0)\}$

Let $\alpha, \beta, \gamma, \delta$ be the scalars such that,

$\alpha(1,2,3) + \beta(0,1,2) + \gamma(1,5,0) + \delta(1,0,0) = 0$

$\Rightarrow (\alpha, 2\alpha, 3\alpha) + (0, \beta, 2\beta) + (\gamma, 5\gamma, 0) + (\delta, 0, 0) = 0$

$\Rightarrow (\alpha+\gamma+\delta, 2\alpha+\beta+5\gamma, 3\alpha+2\beta) = 0.$

$\Rightarrow \alpha+\gamma+\delta = 0, \quad 2\alpha+\beta+5\gamma = 0, \quad 3\alpha+2\beta = 0.$

Given vectors are L.D.

[illegible] L.D or L.I

i) $S = \{(1,0,0), (0,2,0)\}$

ii) $S = \{(0,1,-2), (1,-1,1), (1,2,1)\}$

→ i) Given vectors are

$S = \{(1,0,0), (0,2,0)\}$

Let α, β, γ be the scalars such that,

$\alpha(1,0,0) + \beta(0,2,0) = 0$

$(\alpha,0,0) + (0,2\beta,0) = 0$

$(\alpha, 2\beta, 0) = (0,0,0)$

$\therefore \alpha = 0, \quad 2\beta = 0, \quad 0 = 0.$

$\therefore$ Given vectors are Linearly Independent.

ii) Given vectors are,

$S = \{(0,1,-2), (1,-1,1), (1,2,1)\}$

α, β, γ be the scalars

$\alpha(0,1,-2) + \beta(1,-1,1) + \gamma(1,2,1) = 0$

$(0, \alpha, -2\alpha) + (\beta, -\beta, \beta) + (\gamma, 2\gamma, \gamma) = 0.$

$(\beta+\gamma, \ \alpha-\beta+2\gamma, \ -2\alpha+\beta+\gamma) = 0.$

$\begin{matrix} 0 & 1 & 1 \\ 1 & -1 & 2 \\ -2 & 1 & 1 \end{matrix}$

$\therefore \beta+\gamma = 0, \quad \alpha-\beta+2\gamma = 0, \quad -2\alpha+\beta+\gamma = 0.$

$\beta = -\gamma, \quad 0-\gamma+2\gamma = 0, \quad -2\alpha-\gamma+\gamma = 0$

$\gamma = 0 \qquad -2\alpha = 0$

$\alpha = 0$

$\therefore \alpha = 0, \ \beta = 0, \ \gamma = 0.$

$\therefore$ S is L.I.

iii) $S = \{(1,0,0), (1,1,1), (1,2,3)\} \subset R^3$

iv) $S = \{(1,1,0), (1,0,1), (0,1,1)\} \subset R^3$.

→ Given, $S =$

α, β, γ be the scalars such that,

$\alpha(1,0,0) + \beta(1,1,1) + \gamma(1,2,3) = 0$

$(\alpha,0,0) + (\beta,\beta,\beta) + (\gamma, 2\gamma, 3\gamma) = 0$

$(\alpha+\beta+\gamma, \beta+2\gamma, \beta+3\gamma) = (0,0,0)$

$\therefore \alpha+\beta+\gamma = 0, \beta+2\gamma = 0, \beta+3\gamma = 0$

$\alpha+0+0 = 0$ $\beta = -2\gamma$ $-2\gamma+3\gamma = 0$

$\alpha = 0$. $\therefore \beta = 0$ $\gamma = 0$

$\therefore$ S is L. I.

iv) Given $S =$

α, β, γ be the scalars

$\alpha(1,1,0) + \beta(1,0,1) + \gamma(0,1,1) = 0$

$(\alpha,\alpha,0) + (\beta,0,\beta) + (0,\gamma,\gamma) = 0$

$(\alpha+\beta, \alpha+\gamma, \beta+\gamma) = 0$.

$\alpha+\beta = 0$, $\alpha+\gamma = 0$, $\beta+\gamma = 0$

$\alpha = -\beta$ $-\beta+\gamma = 0$

$-\beta = -\gamma$ $\gamma+\gamma = 0 \Rightarrow 2\gamma = 0$

$\therefore \alpha = 0$ $\therefore \beta = 0$ $\gamma = 0$

$\therefore$ S is L.I.

5) In a vector space P of polynomials of vectors $f(x) = 1-x$, $g(x) = x-x^2$, $h(x) = 1-x^2$ are linearly dependent

→ Given vectors
• $f(x) = 1-x$, $g(x) = x-x^2$, $h(x) = 1-x^2 \in P$
where P is vector space of polynomials.

Let α, β, γ be scalars such that,

$\alpha f(x) + \beta \cdot (g(x)) + \gamma \cdot h(x) = 0$
$\Rightarrow \alpha(1-x) + \beta(x-x^2) + \gamma(1-x^2) = 0$

$\Rightarrow \alpha - \alpha x + \beta x - \beta x^2 + \gamma - \gamma x^2 = 0$

$\Rightarrow (-\beta-\gamma)x^2 + (\beta-\alpha)x + (\alpha+\gamma) = 0 \cdot x^2 + 0 \cdot x + 0$

$\Rightarrow -\beta-\gamma = 0$, $\beta-\alpha = 0$, $\alpha+\gamma = 0$.

$-\beta = -\gamma$ $\quad$ $\beta = \alpha$ $\quad$ $\cancel{\beta - \beta = 0}$
$\alpha = -\gamma$

put $\gamma = t$ where t is parameter
$\therefore \alpha = -t$, $\beta = \alpha = -t$
$\therefore \alpha = -t$, $\beta = -t$, $\gamma = t$.

If $t = -1$
$\therefore \alpha = 1$, $\beta = 1$, $\gamma = -1$

$\Rightarrow 1(1-x) + 1(x-x^2) - 1(1-x^2) = 0$
$\Rightarrow (1-x) + (x-x^2) = (1-x^2)$
$\therefore f(x), g(x), h(x)$ are L.D.

Show that the vectors $(0,1,-2), (1,-1,1), (1,2,1)$ are L.I. in R^3.

→ Given vectors,
$(0,1,-2)$, $(1,-1,1)$, $(1,2,1)$
Let α, β, γ be scalars such that,

$\alpha(0,1,-2) + \beta(1,-1,1) + \gamma(1,2,1) = 0$

$\therefore (0, \alpha, -2\alpha) + (\beta, -\beta, \beta) + (\gamma, 2\gamma, \gamma) = 0$

$\therefore (\beta+\gamma, \alpha-\beta+2\gamma, -2\alpha+\beta+\gamma) = 0$

$\therefore \beta+\gamma = 0$, $\alpha-\beta+2\gamma = 0$, $-2\alpha+\beta+\gamma = 0$.

$\beta = -\gamma$, $\alpha+\gamma+2\gamma = 0$
$\alpha = -3\gamma$
$6\gamma - \gamma + \gamma = 0$
$6\gamma = 0$
$\gamma = 0$

$\therefore \beta = 0$ $\alpha = 0$

$\therefore$ Given vectors are L.I.

Note -

1) If $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ are L.I. iff

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \neq 0$$

2) If $(x_1, y_1), (x_2, y_2)$ are L.I. iff

$$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \neq 0$$

3) If vectors u, v, w are L.D. iff $|A| = 0$.
eg. $(1,0,1)$, $(2,0,0)$, $(0,0,3)$

$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 1(0) + 1(0) = 0.$$

$\therefore$ vectors are linearly dependent

4) A non-zero vector is always L.I. i.e. $V \neq 0$
$\therefore$ $\alpha \cdot V = 0$ then $\alpha = 0$.
$\therefore$ $\{V\}$ is L.I.

e.g. $\{2, 3\}$ is L.I.

5) A zero vector is linearly dependent.
eg. $\{(0,0)\}$ is L.D.
i.e. if $\alpha \neq 0$ then $\alpha(0,0) = 0$

6) If $S \subseteq V$ is a set containing zero vector then S is L.D.
eg. $S = \{(2,3), (0,0)\}$ then S is L.D.
since $\alpha(2,3) + \beta(0,0) = 0$
$\Rightarrow (2\alpha, 3\alpha) = 0$
$\Rightarrow \alpha = 0$ and β is any scalar

7) If u, v are L.D. then their exist non-zero scalars, α, β such that $\alpha u + \beta v = 0$.
$\therefore$ $\alpha u = -\beta v$
$\therefore$ $u = \frac{(-\beta)}{\alpha} \cdot v$
$\because$ $\alpha \neq 0$

∴ u is a scalar multiple of v i.e.
if two vectors are L.D. then one of the vector is scalar multiple of other vector.

8) If u, v, w are L.D. then their exist non-zero scalars α, β, γ such that, $\alpha u + \beta v + \gamma w = 0$
$\Rightarrow \alpha u = -\beta v - \gamma w$

$$\therefore u = \left(\frac{-\beta}{\alpha}\right) v + \left(\frac{-\gamma}{\alpha}\right) w$$

$\because \alpha \neq 0$
$\therefore u = av + bw$
∴ u is a linear combination of v and w.

9) If V is n dimensional vector space and $S \subseteq V$ containing more than n elements then S is L.D.
eg. $S = \{(2,3), (1,5), (2,-1)\} \subseteq R^2$ is L.D.

$S = \{(2,3,1), (4,6,8), (1,2,5), (5,9,1)\} \subset R^3$ is L.D.

10) Linear dependence depends not only on the vector space but the field as well
eg. i) $C(R)$, $\alpha, \beta \in R$. & $1, i \in C$.

$\therefore \alpha \cdot 1 + \beta \cdot i = 0$
$\alpha + i\beta = 0 + i0$.
$\alpha = 0$, $\beta = 0$.
$\therefore \{1, i\}$ is L.I.

ii) $C(C)$ $\alpha, \beta \in C$ & $1, i \in C$.
$\alpha(1) + \beta(i) = 0$
$\alpha + i\beta = 0$.

Let $\alpha = 1$ & $\beta = i$

$$1 + i \cdot i = 0.$$

$$1 + (-1) = 0.$$

$\therefore \{1, i\}$ is L.D.

6) Show that the vectors $(1,1,2,4)$, $(2,-1,-5,2)$, $(1,-1,-4,0)$, $(2,1,1,6)$

7) If x, y, z are L.I. in V then $x+y$, $y+z$, $x+z$ are also L.I.

→ Since x, y, z are L.I.

$\therefore$ By def^n there exist scalars a, b, c such that $ax + by + cz = 0$.

$$\Rightarrow a = b = c = 0.$$

Let α, β, γ be scalars such that,

$$\alpha(x+y) + \beta(y+z) + \gamma(x+z) = 0.$$

$$\Rightarrow (\alpha+\gamma)x + (\alpha+\beta)y + (\beta+\gamma)z = 0$$

But x, y, z are L.I.

$$\Rightarrow \alpha + \gamma = 0, \quad \alpha + \beta = 0, \quad \beta + \gamma = 0.$$

$$\alpha = -\beta$$

$-\beta + \gamma = 0$

$-\beta = -\gamma$ $\qquad\qquad\qquad\qquad$ $\gamma + \gamma = 0$

$\therefore \beta = 0$ $\qquad\qquad$ $\therefore \alpha = 0.$ $\qquad\qquad$ $\gamma = 0$

$\therefore (x+y), (y+z), (z+x)$ are L.I.

8) If x, y, z are L.I. in V then $x+2y+z$, $x+z$, $x+y-3z$ are L.I.

→ Given vectors, $x+2y+z$, $x+z$, $x+y-3z$
Let α, β, γ be scalars such that,

$$\alpha(x+2y+z) + \beta(x+z) + \gamma(x+y-3z) = 0$$

$$(\alpha x + \beta x + \gamma x, 2\alpha y + \gamma y, \alpha z + \beta z - 3\gamma z) = 0.$$

$$((\alpha+\beta+\gamma)x + (2\alpha+\gamma)y + (\alpha+\beta-3\gamma)z) = 0.$$

since x, y, z are L.I.

$\therefore$ ① $\alpha+\beta+\gamma = 0$, $2\alpha+\gamma = 0$, $\alpha+\beta-3\gamma = 0$.

$\gamma = -2\alpha$

$\alpha + \beta - 2\alpha = 0$ $\quad$ $\beta+\beta-6$

$\beta - \alpha = 0$

$\beta = \alpha$

* Basis of a Vector Space

Let $V(F)$ be a vector space. A subset S of is called a basis of V if S consist of linearl independent elements and $V = L(S)$ i.e. S spans V or a subset S of a vector space V is called a basis of V if
1) S is linearly independend
2) $V = L(S)$.

* Dimension of Vector Space

Number of elements in basis of a vector space $V(F)$ is called dimension of $V(F)$.
It is denoted by dim V

eg. 1) $S = \{(1,0), (0,1)\}$ is basis for R^2.

→ Let α, β be scalars such that,

$$\alpha(1,0) + \beta(0,1) = 0$$

$$\Rightarrow (\alpha, 0) + (0, \beta) = 0$$

$$\Rightarrow (\alpha, \beta) = 0$$

$$\Rightarrow \alpha = 0, \beta = 0.$$

$\therefore S = \{(1,0), (0,1)\}$ is L.I. ——— ①.

By defn, $L(S) = \{\alpha(1,0) + \beta(0,1) / \alpha, \beta \in R\}$

$$= \{(\alpha, \beta) / \alpha, \beta \in R\}$$

$$= R^2. \text{ ——— ②}.$$

$\therefore$ S is a basis for R^2.
This basis is called standard basis for R^2.
$\therefore$ Dimension of $R^2 = 2$

2) $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ is a standard basis for R^3.
$\therefore$ Dimension of $R^3 = 3$.

3) $\{(1,0,0,\ldots 0), (0,1,0,\ldots 0), \ldots\ldots, (0,0,\ldots\ldots 1)\}$ is

a standard basis for R^n

$\therefore$ Dimension of $R^n = n$.

4) Show that $\{(2,3), (0,5)\}$ is a basis for R^2.

$\rightarrow$ i) Let α, β be scalars such that,

$\alpha(2,3) + \beta(0,5) = 0$

$\Rightarrow (2\alpha, 3\alpha) + (0, 5\beta) = 0$

$\Rightarrow (2\alpha, 3\alpha + 5\beta) = (0,0)$

$\Rightarrow 2\alpha = 0, \quad 3\alpha + 5\beta = 0$

$\Rightarrow \alpha = 0, \quad \beta = 0.$

$\therefore S = \{(2,3), (0,5)\}$ is L.I.

ii) By def^n

$L(S) = \{\alpha(2,3) + \beta(0,5) \,/\, \alpha, \beta \in R\}$

$= \{(2\alpha, 3\alpha + 5\beta) \,/\, \alpha, \beta \in R\}$

$= \{(x, y) \,/\, x = 2\alpha, \; y = 3\alpha + 5\beta\}$

$= R^2$

$\therefore$ S is basis for R^2.

From 1, 2 $\{(2,3), (0,5)\}$ is basis for R^2.

5) Show that $S = \{(1,1,0), (1,0,0), (0,1,1)\}$ is basis for R^3.

→ Let α, β, γ be scalars such that

$\alpha(1,1,0) + \beta(1,0,0) + \gamma(0,1,1) = 0$

$\Rightarrow (\alpha, \alpha, 0) + (\beta, 0, 0) + (0, \gamma, \gamma) = 0$

$\Rightarrow (\alpha+\beta, \alpha+\gamma, \gamma) = (0,0,0)$

$\Rightarrow \alpha+\beta = 0, \alpha+\gamma = 0, \gamma = 0$

$\Rightarrow \beta = 0, \alpha = 0, \gamma = 0$

$\therefore S = \{(1,1,0), (1,0,0), (0,1,1)\}$ is L.I. —— ①

By def^n

$L(S) = \{\alpha(1,1,0) + \beta(1,0,0) + \gamma(0,1,1) / \alpha, \beta, \gamma \in R\}$

$= \{(\alpha+\beta, \alpha+\gamma, \gamma) / \alpha, \beta, \gamma \in R\}$

$L(S) = R^3$ —— ②

$\therefore$ from ① & ②

$\{(1,1,0), (1,0,0), (0,1,1)\}$ is basis for R^3.

* Theorem

If $S = \{v_1, v_2, \ldots v_n\}$ is a basis of V then every element of V can be expressed uniquely as a linear combination of $v_1, v_2, \ldots v_n$

→ Let $V(F)$ be a vector space.

Let $S = \{v_1, v_2, \ldots, v_n\}$ be a basis of V

$\therefore$ By def^n 1) S is L.I. & 2) $V = L(S)$.

since $V = L(S)$

$\therefore$ Every element in V can be expressed as a linear combination of elements in S.

i.e. $v_1, v_2, v_3, \dots v_n$

Uniqueness -

Let $v \in V$

$\therefore v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \dots + \alpha_n v_n \quad , \alpha_i \in F$ — ①

Suppose $v = \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n \quad , \beta_i \in F$ — ②

$\therefore$ From ① & ②

$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n$

$\therefore (\alpha_1 - \beta_1) v_1 + (\alpha_2 - \beta_2) v_2 + \dots (\alpha_n - \beta_n) v_n = 0.$

But $v_1, v_2 \dots v_n$ are L.I.

$\therefore \alpha_i - \beta_i = 0 \quad \forall \ i$

$\Rightarrow \alpha_i = \beta_i \quad \forall \ i$

$\therefore v$ can be uniquely expressed as a linear combination of $v_1, v_2, \dots v_n$

$\therefore$ Every ~~eli~~ element in V can be expressed uniquely as a linear combination of $v_1, v_2, \dots v_n$

* Theorem

Suppose S is a finite subset of a vector space V such that $V = L(S)$ then there exist a subset of S which is basis of V

→ If S consist of L.I. element then S itself forms basis of V and therefore we have nothing to prove.

Now let T be a subset of S such that T spans V i.e. $V = L(T)$

and T is such minimal subset of S.

Suppose $T = \{v_1, v_2, \ldots v_n\}$

Now show that T is L.I.

Let $\alpha_1, \alpha_2, \ldots \alpha_n$ be n scalars such that

$\alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_n v_n = 0$

suppose $\alpha_i \neq 0$ for some i

without any loss of generality we can take $\alpha_1 \neq 0$

$\therefore \alpha_1^{-1}$ exist.

~~since $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$~~

premultiplying by α_1^{-1},

$\alpha_1^{-1}(\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n) = 0$

$\Rightarrow v_1 + \alpha_1^{-1}\alpha_2 v_2 + \alpha_1^{-1}\alpha_3 v_3 + \ldots + \alpha_1^{-1}\alpha_n v_n = 0.$

$\Rightarrow v_1 = (-\alpha_1^{-1}\alpha_2) v_2 + (-\alpha_1^{-1}\alpha_3) v_3 + \ldots + (-\alpha_1^{-1}\alpha_n) v_n$

$\therefore v_1 = \beta_2 v_2 + \beta_3 v_3 + \ldots + \beta_n v_n$ —— ①

Let $v \in V = L(T)$

$\therefore v = \gamma_1 v_1 + \gamma_2 v_2 + \ldots + \gamma_n v_n$

$\therefore v = \gamma_1(\beta_2 v_2 + \beta_3 v_3 + \ldots \beta_n v_n) + \gamma_2 v_2 + \gamma_3 v_3 + \ldots \gamma_n$

$= (\gamma_1\beta_2 + \gamma_2) v_2 + (\gamma_1\beta_3 + \gamma_3) v_3 + \ldots + (\gamma_1\beta_n + \gamma_n) v_n$

$\therefore$ any element of V is a linear combination of $v_2, v_3, \dots v_n$

$\therefore$ The set $\{v_2, v_3, \dots v_n\}$ spans V which contradicts our choose of T.

$\therefore$ Our supposition is wrong.

i.e. $\alpha_i \neq 0$ is wrong.

$\therefore \alpha_i = 0 \quad \forall \; i$

$\therefore v_1, v_2, \dots v_n$ are L.I. vectors.

$\therefore T$ is linearly independent and $V = L(T)$

$\therefore T$ is basis of V.

$\therefore$ there exist a subset T of S which is basis of V

Def^n - A F.D.V.S. 'V' is said to have dimension n if n is number of elements in any basis of V. It is denoted by $\dim V$ or $\dim_F V$.

eg. $\dim R^2 = 2$, $\dim R^3 = 3$, $\dim R^n = n$
$\dim C(R) = 2$.

Note -

1) A F.D.V.S. V has dimension 'n' iff n is the maximum number of Linearly independent vectors in any subset of V

2) If $\dim V = n$ then any $n+1$ vectors in V are L.D

3) If $\dim V = n$ and $\{v_1, v_2, \dots v_m\}$ are L.I. them

$m \leq n$

→ 6) Given vectors are $(1,1,2,4)$, $(2,-1,-5,2)$, $(1,-1,-4,0)$, $(2,1,1,6)$

Let $\alpha, \beta, \gamma, \delta$ be scalars such that,

$$\alpha(1,1,2,4)+\beta(2,-1,-5,2)+\gamma(1,-1,-4,0)+\delta(2,1,1,6)=0$$

$$\Rightarrow (\alpha+2\beta+\gamma+2\delta,\ \alpha-\beta-\gamma+\delta,\ 2\alpha-5\beta-4\gamma+\delta,\ 4\alpha+2\beta+6\delta)=0.$$

$$\Rightarrow \alpha+2\beta+\gamma+2\delta=0, \quad \alpha-\beta-\gamma+\delta=0,$$
$$2\alpha-5\beta-4\gamma+\delta=0, \quad 4\alpha+2\beta+6\delta=0$$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & 6 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - 4R_1$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & -9 & -6 & -3 \\ 0 & -6 & -4 & -2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & -3 & -2 & -1 \\ 0 & -3 & -2 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\alpha + 2\beta + \gamma + 2\delta = 0$

& $-3\beta - 2\gamma - \delta = 0 \Rightarrow 3\beta + 2\gamma + \delta = 0$

* Theorem -

If V is F.D.V.S. and $\{v_1, v_2, v_3, \ldots, v_r\}$ is the linearly independent subset of V then it can be extended to form the basis of V.

→ If $\{v_1, v_2, \ldots, v_r\}$ spans V then it itself form a basis of V

∴ There is nothing to prove

Let $S = \{v_1, v_2, \ldots, v_r, v_{r+1}, \ldots, v_n\}$

be the maximal L.I. subset of V.

To show that S is basis of V for which to show that S spans V. i.e. prove that $V = L(S)$ i.e. Prove that every element in V is a linear combination of elements in S.

Let $v \in V$ be any element then

$T = \{v_1, v_2, \ldots, v_n, v\}$ is L.D.

since by choose of S.

$\Rightarrow$ There exist $\alpha_1, \alpha_2, \ldots \alpha_n, \alpha \in F$ (Not all zero) such that,

$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n + \alpha v = 0.$

To prove that $\alpha \neq 0$

But suppose $\alpha = 0$

$\therefore \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0.$

but $v_1, v_2, \ldots v_n$ are L.I.

$\therefore \alpha_i = 0 \quad \forall i$

which is contradict to $\alpha = 0$.

$\therefore \alpha \neq 0$

$\therefore \alpha^{-1}$ exist.

$\therefore \alpha^{-1}(\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n + \alpha v) = 0.$

$\therefore (\alpha^{-1}\alpha_1) v_1 + (\alpha^{-1}\alpha_2) v_2 + \ldots + (\alpha^{-1}\alpha_n) v_n + v = 0.$

$\therefore v = (-\alpha^{-1}\alpha_1) v_1 + (-\alpha^{-1}\alpha_2) v_2 + \ldots + (-\alpha^{-1}\alpha_n) v_n$

$= \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n$

$\therefore$ Every vector in V can be expressed as a linear combination of $v_1, v_2, \ldots v_n$

$\therefore S$ spans V

$\therefore S$ is a basis set of V.

$\therefore$ Every linearly independent subset of V can be extended to form the basis of V.

* Theorem

If $\dim V = n$ and $S = \{v_1, v_2, \ldots, v_n\}$ spans V then S is a basis of V.

→ Since $\dim V = n$

∴ Any basis of V has n elements.

Given that $S = \{v_1, v_2, \ldots, v_n\}$ and $V = L(S)$

We know that, If S is a finite subset of vector space V such that $V = L(S)$ then there exist a subset of S which is basis of V — ①

∴ from ① a subset of S will be a basis of V but S contains n elements

∴ S itself forms a basis of V

Theorem

If $\dim V = n$ and $S = \{v_1, v_2, \ldots, v_n\}$ is L.I. of V then S is basis of V

→ Since $S = \{v_1, v_2, \ldots, v_n\}$ is L.I.

∴ It can be extended to form a basis of V

Since if V is FDVS and $\{v_1, v_2, \ldots, v_r\}$ is L.I. subset of V then it can be extended to form the basis of V

But $\dim V = n$ i.e. any basis of V has n elements.

∴ S itself form a basis of V

Note -

1) Two F.D.V.S over F are isomorphic iff they have the same dimension.

2) Let W be a subspace of a FDVS V then $\dim W \leq \dim V$ infact $\dim V = \dim W$ iff $V = W$

3) If W is a subspace of V and $W = \{0\}$ then $\dim W = 0$ i.e. dimension of zero space $= 0$.

4) Let W be a subspace of FDVS V then dimension of quotient space i.e. $\dim\left(\frac{V}{W}\right) = \dim V - \dim W$

5) If A and B are two subspaces of FDVS 'V' then $\dim(A+B) = \dim A + \dim B - \dim A \cap B$. and $\dim(A \oplus B) = \dim A + \dim B$

6) W' is called complement of W

7) Every subspace of FDVS has a complement.

8) If W' is complement of W in V then ~~di~~ $\dim W' = \dim V - \dim W$.

2. Inner Product Space

Inner Product Space

Def^n - Let V be a vector space over field F where F is field of real numbers or complex numbers). Suppose for any two vectors $u, v \in V$ there exist an element $(u, v) \in F$ such that,

1) $(u, v) = \overline{(v, u)}$ (i.e. complex conjugate of (v,u)
2) $(u, u) \geq 0$ and $(u, u) = 0$ iff $u = 0$.
3) $(\alpha u + \beta v, w) = \alpha(u, w) + \beta(v, w)$.

   OR $(u+v, w) = (u, w) + (v, w)$ & $(\alpha u, w) = \alpha(u, w)$

   $\forall$ $u, v, w \in V$ and $\forall$ $\alpha, \beta \in F$.

This product is called an inner product. The product satisfying ①, ②, ③ is called an inner product and vector space V together with an inner product is called an inner product space i.e. Inner product space is a vector space over the field of real or complex numbers with an inner product function.

Note -

1) If F = field of real numbers then the inner product is called dot product.

2) Inner product space over real field is called Euclidean space

3) Inner product space over complex field is called unitary space.

1) Show that in an inner product space V, ∀
$(u, \alpha v + \beta w) = \bar{\alpha}(u, v) + \bar{\beta}(u, w)$

→ Let $u, v, w \in V$ and $\alpha, \beta \in F$

$(u, \alpha u + \beta w) = \overline{(\alpha v + \beta w, u)}$

$= \overline{\alpha (v, u) + \beta (w, u)}$ $\quad \left[(\alpha u + \beta v, w) = \alpha (u, w) + \beta (v, w) \right]$

$= \overline{\alpha (v, u)} + \overline{\beta (w, u)}$

$= \bar{\alpha} \, (\overline{v, u}) + \bar{\beta} \, (\overline{w, u})$

$= \bar{\alpha} (u, v) + \bar{\beta} (u, w)$

2) Let $R^2(R)$ be a vector space w.r.t. usual vector addition and scalar multiplication i.e. i.e. $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $\alpha (x_1, y_1) = (\alpha x_1, \alpha y_1)$. Define product $u = (x_1, y_1)$, $v = (x_2, y_2) \in R^2$ such that,

$(u, v) = x_1 x_2 + y_1 y_2$. Show that $R^2(R)$ is an inner product space w.r.t. given product.

→ Let $R^2(R) = \{ (x, y) / x, y \in R \}$.
w.r.t. usual vector addition and scalar multiplication.

1) $(u, v) = x_1 x_2 + y_1 y_2$.

Now $\overline{(v,u)} = \overline{(x_2x_1 + y_2y_1)}$

$= \overline{x_2x_1} + \overline{y_2y_1}$

$= x_2x_1 + y_2y_1$ $\because u=\overline{u}$ when u = real no

$= x_1x_2 + y_1y_2$

$= (u,v)$

2) $(u,u) = x_1x_1 + y_1y_1$

$= x_1^2 + y_1^2 \geq 0.$

if $(u,u)=0 \iff x_1^2+y_1^2=0$

$\iff x_1^2=0,\ y_1^2=0$

$\iff x_1=0,\ y_1=0$

$\iff u=(x_1,y_1)=(0,0)=\overline{0}$

$(u,u)=0 \iff u=0.$

3) Let $\alpha, \beta \in R$.

Let $u=(x_1,y_1)$, $v=(x_2,y_2)$, $w=(x_3,y_3) \in R^2$

$\therefore \alpha u + \beta v = \alpha(x_1,y_1) + \beta(x_2,y_2)$

$= (\alpha x_1, \alpha y_1) + (\beta x_2, \beta y_2)$

$= (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)$

$\therefore \langle \alpha u + \beta v, w \rangle = \langle (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2), (x_3, y_3) \rangle$

$= (\alpha x_1 + \beta x_2)x_3 + (\alpha y_1 + \beta y_2)y_3$

$= \alpha x_1x_3 + \beta x_2x_3 + \alpha y_1y_3 + \beta y_2y_3$

$= \alpha x_1x_3 + \alpha y_1y_3 + \beta x_2x_3 + \beta y_2y_3$

$$= \alpha(x_1x_3 + y_1y_3) + \beta(x_2x_3 + y_2y_3)$$

$$= \alpha(u,\omega) + \beta(v,\omega)$$

$\therefore$ Given product is an inner product
$\therefore$ $R^2(R)$ is an inner product space.

3) Let $c^2(c) = \{(x,y) / x, y \in C\}$ i.e.
$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$
$\alpha(x_1, y_1) = (\alpha x_1, \alpha y_1)$
define product $u = (x_1, y_1)$, $v = (x_2, y_2) \in c^2$
such that $\langle u, v \rangle = x_1 \overline{x_2} + y_1 \cdot \overline{y_2}$ show that $c^2(c)$ is an inner product space

→ Let $c^2(c) = \{(x,y) / x, y \in C\}$
w.r.t. usual vector addition and scalar multiplication

1) $(u,v) = x_1\overline{x_2} + y_1\overline{y_2}$

$$\overline{\langle v, u\rangle} = \overline{x_2\overline{x_1} + y_2\overline{y_1}}$$

$$= \overline{x_2\overline{x_1}} + \overline{y_2\overline{y_1}}$$

$$= \overline{x_2}\, x_1 + \overline{y_2}\, y_1$$

$$= x_1\overline{x_2} + y_1\overline{y_2}$$
$$= \langle u, v\rangle$$

2) $\langle u, u\rangle = x_1\overline{x_1} + y_1\overline{y_1}$
$$= |x_1|^2 + |y_1|^2 \geq 0$$

if $\langle u,u\rangle = 0 \iff |x_1|^2 + |y_1|^2 = 0$

$\iff |x_1|^2 = 0$ or $|y_1^2| = 0$

$\iff x_1 = 0 \quad , \; y_1 = 0$

$\iff u = (x_1, y_1)$

$= (0, 0)$

$= 0.$

Let $\alpha, \beta \in C$ and $u = (x_1, y_1)$, $v = (x_2, y_2)$ & $\omega = (x_3, y_3) \in C^2$

$$\alpha u + \beta v = \alpha(x_1, y_1) + \beta(x_2, y_2)$$
$$= (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)$$

$$\langle \alpha u + \beta v, \omega\rangle = (\alpha x_1 + \beta x_2)\overline{x_3} + (\alpha y_1 + \beta y_2)\overline{y_3}$$

$$= \alpha x_1 \overline{x_3} + \beta x_2 \overline{x_3} + \alpha y_1 \overline{y_3} + \beta y_2 \overline{y_3}$$

$$= \alpha(x_1\overline{x_3} + y_1\overline{y_3}) + \beta(x_2\overline{x_3} + y_2\overline{y_3})$$

$$= \alpha(u, \omega) + \beta(v, \omega).$$

$\therefore$ Given product is an inner product.

$\therefore$ $C^2(C)$ is an inner product space.

Note -

These inner product are called standard inner product.

4) Let $V=R^2$ $u=(\alpha_1,\alpha_2)$, $v=(\beta_1,\beta_2)$. Define product of $(u,v)=\alpha_1\beta_1-\alpha_2\beta_1-\alpha_1\beta_2+4\alpha_2\beta_2$ show that R^2 is an inner product space w.r.t. given product.

→ Let $R^2=\{(x,y)\ |x,y\in R\}$

$$(u,v)=\alpha_1\beta_1-\alpha_2\beta_1-\alpha_1\beta_2+4\alpha_2\beta_2$$

$$\overline{\langle v,u\rangle}=\overline{\beta_1\alpha_1-\beta_2\alpha_1-\beta_1\alpha_2+4\beta_2\alpha_2}$$

$$=\overline{\beta_1\alpha_1}-\overline{\beta_2\alpha_1}-\overline{\beta_1\alpha_2}+\overline{4\beta_2\alpha_2}$$

$$=\beta_1\alpha_1-\beta_2\alpha_1-\beta_1\alpha_2+4\beta_2\alpha_2$$

$$=\alpha_1\beta_1-\alpha_1\beta_2-\alpha_2\beta_1+4\alpha_2\beta_2$$

$$=\langle u,v\rangle$$

ii) $\langle u,u\rangle=\alpha_1\alpha_1-\alpha_2\alpha_1-\alpha_1\alpha_2+4\alpha_2\alpha_2$

$$=\alpha_1^2-2\alpha_1\alpha_2+4\alpha_2^2$$

$$=(\alpha_1^2-2\alpha_1\alpha_2+\alpha_2^2)+3\alpha_2^2$$

$$=(\alpha_1-\alpha_2)^2+3\alpha_2^2\quad\geq 0.$$

$\langle u,u\rangle=0\iff(\alpha_1-\alpha_2)^2+3\alpha_2^2=0$

$\iff(\alpha_1-\alpha_2)^2=0\quad,\quad\alpha_2^2=0$

$\iff\alpha_1-\alpha_2=0\quad,\quad\alpha_2=0$

$\iff\alpha_1=0\quad,\alpha_2=0.$

$$\Leftrightarrow u = (\alpha_1, \alpha_2)$$
$$= (0,0) = 0.$$

Let $\alpha, \beta \in R$, let $u, v, w = (\gamma_1, \gamma_2) \in R^2$

$$\alpha u + \beta v = \alpha(\alpha_1, \alpha_2) + \beta(\beta_1, \beta_2)$$
$$= (\alpha\alpha_1 + \beta\beta_1, \alpha\alpha_2 + \beta\beta_2)$$

$$\langle \alpha u + \beta v, w \rangle = (\alpha\alpha_1 + \beta\beta_1)\gamma_1 - (\alpha\alpha_2 + \beta\beta_2)\gamma_1 - (\alpha\alpha_1 + \beta\beta_1)\gamma_2 + 4(\alpha\alpha_2 + \beta\beta_2)\gamma_2$$

$$= \alpha\alpha_1\gamma_1 + \beta\beta_1\gamma_1 - \alpha\alpha_2\gamma_1 - \beta\beta_2\gamma_1 - \alpha\alpha_1\gamma_2 - \beta\beta_1\gamma_2 + 4\alpha\alpha_2\gamma_2 + 4\beta\beta_2\gamma_2$$

$$= \alpha(\alpha_1\gamma_1 - \alpha_2\gamma_1 - \alpha_1\gamma_2 + 4\alpha_2\gamma_2)$$
$$+ \beta(\beta_1\gamma_1 - \beta_2\gamma_1 - \beta_1\gamma_2 + 4\beta_2\gamma_2)$$

$$= \alpha(u, w) + \beta(v, w)$$

$\therefore$ Given product is inner product ~~space~~.
$\therefore R^2$ is an inner space

Let V be an inner product space then show that
i) $\langle 0, v \rangle = 0 \quad \forall\ v \in V$
ii) $\langle v, v \rangle = 0 \quad \forall\ v \in V \Rightarrow v = 0.$

→ Let V be an inner product space.
Let $v \in V$

$$\langle 0, v\rangle = \langle 0\cdot 0, v\rangle \quad \because \{\langle \alpha u, v\rangle = \alpha\langle u, v\rangle\}$$
$$= 0\langle 0, v\rangle$$
$$= 0$$

Let $u, v \in V$

$\langle u, v\rangle = 0 \quad \forall\ v \in V$

$\Rightarrow \langle u, u\rangle = 0 \quad \because$ V is an inner product space.

$\Rightarrow u = 0.$

* Norm of a Vector

Def^n - Let V be an inner product space. Let $v \in V$ then norm of v (or length of v) is defined as $\sqrt{\langle v, v\rangle}$ and it is denoted by $\|v\|$

i.e. $\|v\| = \sqrt{\langle v, v\rangle}$

eg. - 1) In an inner product space R^3

$\|(2, -1, 0)\| =$

$$\|(2, -1, 0)\| = \sqrt{\langle (2, -1, 0), (2, -1, 0)\rangle}$$
$$= \sqrt{4 + 1 + 0}$$
$$= \sqrt{5}.$$

2) $\|\alpha v\| = |\alpha|\,\|v\|$

$$\|\alpha v\|^2 = \langle \alpha v, \alpha v\rangle$$

$$= \alpha \langle v, \alpha v \rangle$$

$$= \alpha \overline{\langle \alpha v, v \rangle}$$

$$= \alpha \overline{\alpha \langle v, v \rangle}$$

$$= \alpha \cdot \bar{\alpha} \overline{\langle v, v \rangle}$$

$$= \alpha \bar{\alpha} \langle v, v \rangle$$

$$\|\alpha v\|^2 = |\alpha|^2 \|v\|^2$$

$$\|\alpha v\| = |\alpha| \cdot \|v\|$$

OR

$$\langle \alpha u + \beta v, \omega \rangle = \alpha \langle u, \omega \rangle + \beta \langle v, \omega \rangle$$

If $\beta = 0$

$$\langle \alpha u, \omega \rangle = \alpha \langle u, \omega \rangle$$

$$\langle u, \alpha v + \beta \omega \rangle = \bar{\alpha} \langle u, v \rangle + \bar{\beta} \langle u, \omega \rangle$$

If $\beta = 0$

$$\langle u, \alpha v \rangle = \bar{\alpha} \langle u, v \rangle.$$

$$\|\alpha v\|^2 = \langle \alpha v, \alpha v \rangle$$

$$= \alpha \langle v, \alpha v \rangle$$

$$= \alpha \overline{\langle \alpha v, v \rangle}$$

$$= \alpha \bar{\alpha} \langle v, v \rangle$$

$$= |\alpha|^2 \|v\|^2$$

$$\|\alpha v\| = |\alpha| \, \|v\|$$

* Cauchy Schwarz Inequality -

Statement - Let V be an inner product space then $|\langle u,v\rangle| \leq \|u\| \cdot \|v\| \quad \forall\ u,v \in V$.

→ Case I -

If $u=0$ then
$\langle u,v\rangle = \langle 0,v\rangle = 0$.

and $\|u\| = \sqrt{\langle u,u\rangle} = \sqrt{\langle 0,0\rangle} = 0$.

$\therefore$ L.H.S. = R.H.S.

Case II -

Let $u \neq 0$
$\therefore \|u\| \neq 0$.

Let $\omega = v - \dfrac{\langle v,u\rangle}{\|u\|^2} \cdot u$.

$$\therefore \langle \omega,\omega\rangle = \left\langle v - \frac{\langle v,u\rangle}{\|u\|^2}\cdot u \ ,\ v - \frac{\langle v,u\rangle}{\|u\|^2}\cdot u \right\rangle$$

$$= \langle v,v\rangle - \frac{\overline{\langle v,u\rangle}}{\|u\|^2}\langle v,u\rangle - \frac{\langle v,u\rangle}{\|u\|^2}\langle u,v\rangle + \frac{\langle v,u\rangle}{\|u\|^2}\,\frac{\overline{\langle v,u\rangle}}{\|u\|^2}\,\langle u,u\rangle$$

$$= \|v\|^2 - \frac{\langle u,v\rangle\overline{\langle u,v\rangle}}{\|u\|^2} - \frac{\langle v,u\rangle\langle v,u\rangle}{\|u\|^2} + \frac{\langle v,u\rangle \ldots}{\|u\|^2}$$

$$\langle w, w \rangle = \|v\|^2 - \frac{|\langle u, v\rangle|^2}{\|u\|^2}$$

i.e. $\|v\|^2 - \dfrac{|\langle u, v\rangle|^2}{\|u\|^2} = \langle w, w\rangle \geq 0.$

$$\|v\|^2 \|u\|^2 - |\langle u, v\rangle|^2 \geq 0$$

$$\|u\|^2 \cdot \|v\|^2 \geq |\langle u, v\rangle|^2$$

$$\Rightarrow |\langle u, v\rangle|^2 \leq \|u\|^2 \cdot \|v\|^2$$

$$\Rightarrow |\langle u, v\rangle| \leq \|u\| \cdot \|v\|$$

* Triangle Inequality and Parellelogram law -

Statement - Let V be an inner product space then

i) $\|x+y\| \leq \|x\| + \|y\|$

ii) $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

→ Let V be an inner product space. Let $x, y \in V$

$$\|x+y\|^2 = \langle x+y, x+y\rangle$$
$$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$$
$$= \|x\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} + \|y\|^2$$

$$= \|x\|^2 + 2\,\mathrm{Re}\langle x, y\rangle + \|y\|^2$$

$$(\because z + \bar{z} = 2\mathrm{Re}(z))$$

$$\leq \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2$$

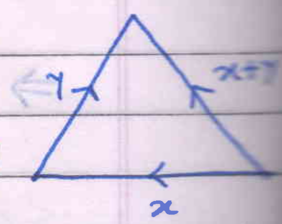
$$(\mathrm{Re}(z) \leq |z|)$$

By Cauchy Schwarz Inequality,

$$\leq \|x\|^2 + 2\|x\|\cdot\|y\| + \|y\|^2$$

$$\|x+y\|^2 \leq (\|x\| + \|y\|)^2$$

$$\|x+y\| \leq \|x\| + \|y\| \quad \text{——— ①}$$



This is called Triangle Inequality.

$\because \|x\| + \|y\| =$ sum of the length of two sides of triangle.

$\|x+y\| =$ length of third side of triangle.

eqⁿ ① shows that third side is less than or equal to the sum of the two sides of triangle.

ii) Let $x, y \in V$

$$\|x+y\|^2 = \langle x+y, x+y\rangle$$

$$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$$

$$= \|x\|^2 + \langle x, y\rangle + \langle x, y\rangle + \|y\|^2$$

$$\|x+y\|^2 = \|x\|^2 + 2\,\text{Re}\langle x, y\rangle + \|y\|^2 \quad \text{--- (2)}$$

$$\|x-y\|^2 = \langle x-y, x-y\rangle$$

$$= \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle$$

$$= \|x\|^2 - \langle x, y\rangle - \langle x, y\rangle + \|y\|^2$$

$$= \|x\|^2 - \left[\langle x, y\rangle + \langle x, y\rangle\right] + \|y\|^2$$

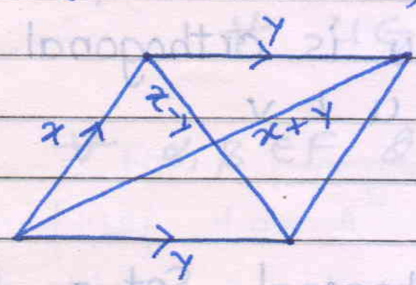
$$= \|x\|^2 - 2\,\text{Re}\langle x, y\rangle + \|y\|^2 \quad \text{--- (3)}$$

Adding, (2) and (3)

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

$$= 2\left(\|x\|^2 + \|y\|^2\right) \quad \text{--- (4)}$$

From (4) we say that sum of the squares of lengths of diagonals of parallelogram is equal to twice the sum of squares of sides of a parallelogram.



$\therefore$ eq^n (4) is called Parallelogram law.

* Orthogonality

Orthogonal Vectors

<u>Def^n</u> - Let V be an inner product space. Two vectors u and v is said to be orthogonal if $\langle u, v\rangle = 0$ iff $\langle v, u\rangle = 0$

Note -

1) u is orthogonal to v iff v is orthogonal to u

2) Inner product $\langle 0, v\rangle = 0 \quad \forall \; v \in V$
$\therefore$ 0 is orthogonal to every vector in V.

3) If $u \in V$ is orthogonal to every vector in V
i.e. $\langle u, v\rangle = 0 \;, \forall \; v \in V$
$\Rightarrow \langle u, u\rangle = 0$
$\Rightarrow u = 0.$
$\therefore$ 0 vector is only vector in V which is orthogonal to all vectors in V.

4) u is orthogonal to v is denoted by '$\perp$'
i.e. $u \perp v$

Orthogonal Set -

Def^n - Let V be an inner product space and W be the subspace of V define
$W^{\perp} = \{ v \in V / \langle u, v\rangle = 0, \; \forall \; u \in W \}$

then W is called Orthogonal Set. It is denoted by $W^{\perp}$

eg.- If W is subspace of an inner product space V then show that $W^{\perp}$ is also a subspace.

→ Given that W is a subspace of an inner product space V

By def^n $W^{\perp} = \{ v \in V / \langle u, v \rangle = 0, \forall u \in W \}$.

since $0 \in V$

and $\langle 0, u \rangle = 0 \quad \forall u \in W$

i.e. $0 \perp u, \forall u \in W$

$\therefore 0 \in W^{\perp} \subseteq V$

$\therefore W^{\perp}$ is non-empty subset of V ——— ①

Let $\alpha, \beta \in F \qquad u_1, u_2 \in W^{\perp}$

$\langle u_1, u \rangle = 0 \quad , \quad \langle u_2, u \rangle = 0.$

$$\therefore \langle \alpha u_1 + \beta u_2, u \rangle = \alpha \langle u_1, u \rangle + \beta \langle u_2, u \rangle$$
$$= \alpha \cdot 0 + \beta \cdot 0$$
$$= 0 \quad \forall u \in W$$

$\therefore \alpha u_1 + \beta u_2 \in W^{\perp} \qquad \forall \alpha, \beta \in F \;\&\; u_1, u_2 \in W$ ——— ②

∴ From ① and ②

$W^{\perp}$ is a subspace of V

Note - $W^{\perp}$ is called orthognal compliment of W.

$\therefore V = W \oplus W^{\perp}$

* Pythagoras Theorem -

Let V be an inner product space. Let $x, y \in V$ such that $x \perp y$ then show that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

→ Let V be an inner product space.
Let $x, y \in V$ such that $x \perp y$
$\Rightarrow \langle x, y \rangle = 0 = \langle y, x \rangle$ —— ①

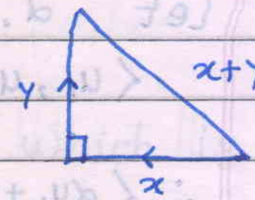
consider

$$\|x+y\|^2 = \langle x+y, x+y \rangle$$

$$= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$$

$$= \|x\|^2 + 0 + 0 + \|y\|^2 \text{ —— from ①}$$

$$= \|x\|^2 + \|y\|^2.$$



∴ This is called Pythagoras Theorem

Orthogonal Set -

A set $\{u_i\}$ of vectors in an inner product space V is said to be orthogonal if
$\langle u_i, u_j \rangle = 0 \quad \forall \; i \neq j$

eg. 1) $\{(1,-2), (2,1)\} \subset R^2$

$$\langle (1,-2), (2,1) \rangle = 1(2) + (-2)(1)$$
$$= 2-2$$
$$= 0.$$

$\therefore (1,-2) \perp (2,1)$

$\therefore \{(1,-2), (2,1)\}$ is orthogonal.

Orthonormal Set -

Def^n - A set $\{u_i\}$ of vectors in an inner product space V is said to be orthonormal

if 1) $\langle u_i, u_j \rangle = 0 \quad \forall \; i \neq j$

i.e $u_i \perp u_j \quad \forall \; i \neq j$

2) $\langle u_i, u_i \rangle = 1 \quad \forall \; i$

i.e. $\|u_i\| = 1 \quad \forall \; i$

eg.- 1) $\{(1,0,0), (0,2,0), (0,0,-3)\} \subset R^3$. Let

$u = 1,0,0$, $v = (0,2,0)$, $w = (0,0,-3)$

Inner product space

→ Let $u = (1,0,0)$, $v = (0,2,0)$, $w = (0,0,-3)$

$\langle u, v \rangle = \langle (1,0,0), (0,2,0) \rangle$

$= 0$.

$\langle u, w \rangle = \langle (1,0,0), (0,0,-3) \rangle$

$= 0$

$\langle v, w \rangle = \langle (0,2,0), (0,0,-3) \rangle$

$= 0$.

$\therefore \{u, v, w\}$ is orthogonal.

$$\|u\| = \sqrt{\langle u,u \rangle} = \sqrt{\langle 1,0,0 \rangle ; \langle 1,0,0 \rangle}$$

$$= \sqrt{1}$$

$$= 1.$$

$$\|v\| = \sqrt{\langle v,v \rangle} = \sqrt{\langle 0,2,0 \rangle, \langle 0,2,0 \rangle}$$

$$= \sqrt{0+4+0}$$

$$= 2$$

$$\|w\| = \sqrt{\langle w,w \rangle} = \sqrt{\langle 0,0,-3 \rangle \ \langle 0,0,-3 \rangle}$$

$$= \sqrt{0+0+9}$$

$$= 3.$$

$\therefore \{u, v, w\}$ is not an orthonormal set.

2) $\{e_1 = (1,0)\ ,\ e_2 = (0,1)\}$

→ Let $e_1 = (1,0)$, $e_2 = (0,1)$

consider

$$\langle e_1, e_2 \rangle = \langle (1,0), (0,1) \rangle$$

$$= \langle 0 + 0 \rangle$$

$$= 0.$$

$$\|e_1\| = \sqrt{\langle e_1, e_1 \rangle} = \sqrt{\langle (1,0), (1,0) \rangle}$$

$$= \sqrt{1}$$

$$= 1.$$

$\|e_2\| = \sqrt{\langle e_2, e_2\rangle} = \sqrt{\langle (0,1),(0,1)\rangle}$

$= \sqrt{0+1}$

$= 1.$

$\therefore \{e_1 = (1,0), e_2 = (0,1)\}$ is orthonormal set.

3) Let V be the real vector space of real polynomials of degree less than or equal to n
i.e.

$V = \{\sum_{i=0}^{n} a_i x^i \,/\, a_i \in R\}$

$V = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \,/\, a_i \in R\}$

define the inner product on V by

$\left(\sum_{i=0}^{n} a_i x^i , \sum_{i=0}^{n} b_i x^i\right) = \sum_{i=0}^{n} a_i b_i$

i.e. $(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n , b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n)$

$= a_0 b_0 + a_1 b_1 + \cdots + a_n b_n.$

then show that $\{1, x, x^2, \ldots, x^n\}$ is an orthonormal set

→ $\langle 1, x\rangle = \langle 1 + 0\cdot x , 0 + 1\cdot x\rangle$

$= 1\cdot 0 + 0\cdot 1 = 0.$

$\langle 1, x^2\rangle = \langle 1 + 0\cdot x + 0\cdot x^2 ; 0 + 0\cdot x + 1\cdot x^2\rangle$

$= 1\cdot 0 + 0\cdot 0 + 0\cdot 1$

$= 0.$

$\therefore \langle x^i, x^j\rangle = 0 \quad \forall\ i \neq j \quad 0 \le i \le n,\ 0 \le j \le n$

$\langle 1, 1\rangle = 1(1) = 1$

$\langle x, x\rangle = \langle 0+1\cdot x, 0+1\cdot x\rangle$

$= 0\cdot 0 + 1\cdot 1$

$= 1$

$\langle x^2, x^2\rangle = \langle 0+0\cdot x+1\cdot x^2, 0+0\cdot x+\phi\, x^2\rangle$

$= 0\cdot 0 + 0\cdot 0 + 1\cdot 1$

$= 1$

$\vdots$

$\langle x^n, x^n\rangle = 1.$

$\therefore \{1, x, x^2, \ldots\ldots x^n\}$ is an orthonormal subset of V.

* Theorem :-

Let S be an orthogonal set of non-zero vectors in an inner product space V then S is linearly independent set.

→ Let $S = \{v_1, v_2, \ldots\ldots v_n\}$ be an orthogonal set of non-zero vectors in an inner product space V.

$\therefore$ By def^n

$\langle v_i, v_j\rangle = 0 \quad \forall\, i \neq j$

To show that S is L.I.

Let $\alpha_1, \alpha_2, \ldots\ldots \alpha_n$ be scalars such that,

$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0.$

Now, $\langle \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n, \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \dots \alpha_n v_n \rangle = 0$

$\Rightarrow |\alpha_1|^2 \|v_1\|^2 + |\alpha_2|^2 \|v_2\|^2 + \dots + |\alpha_n|^2 \|v_n\|^2 = 0$

$\Rightarrow |\alpha_i|^2 \|v_i\|^2 = 0 \quad \forall \; i = 1, 2, \dots n$

but $v_i \neq 0$

$\therefore \|v_i\|^2 \neq 0 \quad \forall i$

$\therefore |\alpha_i|^2 = 0 \quad \forall i \quad \Rightarrow \alpha_i = 0 \; \forall i$ ∴

$\therefore \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$

∴ The set $S = \{v_1, v_2, \dots v_n\}$ is L.I.

Corollary :-

An orthonormal set in an inner product space is L.I.

→ Let S be an orthonormal set in an inner product space V.

∴ S is orthogonal set of non-zero vectors

∴ S is L.I.

* Note :- Every orthogonal set with non-zero vectors in an inner product space V is L.I. but every L.I. set need not be orthogonal

e.g. $\{(1,1), (1,0)\} \subset R^2(R)$

$\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$.

$\therefore \{(1,1),(1,0)\}$ is L.I.
But inner product of $\{(1,1) \& (1,0)\}$

$$\langle (1,1),(1,0)\rangle = 1\cdot 1 + 1\cdot 0 = 1$$

$\therefore (1,1) \not\perp (1,0)$

$\therefore \{(1,1),(1,0)\}$ is not orthogonal.

But we can construct an orthogonal set and hence orthonormal set from given linearly independent set.
The process of constructing an orthogonal set (orthonormal set) from given linearly independent set is called Gram Schmidt orthogonalization (orthonormalization) process.

* Gram Schmidt Orthogonalization Process

Statement - Let V be a non-zero inner product space of diamension n then V has an orthonormal basis.

→ Proof - Given that V is an n-dimensional inner product space.
It is enough to construct an orthogon

basis of V.

For let $S \subseteq V$ be an orthogonal set then $T = \left\{ \frac{x}{\|x\|} \,/\, x \in S \right\}$ is an orthonormal set.

Let $\{v_1, v_2, \ldots v_n\}$ be a basis of V

Step I - Define $\omega_1 = v_1$

Step II - $\omega_2 = v_2 - \frac{\langle v_2, v_1 \rangle}{\|v_1\|^2} \cdot v_1$

$$= v_2 - \frac{\langle v_2, \omega_1 \rangle}{\langle \omega_1, \omega_1 \rangle} \cdot \omega_1$$

Now consider $\langle \omega_2, \omega_1 \rangle = \left\langle v_2 - \frac{\langle v_2, \omega_1 \rangle}{\langle \omega_1, \omega_1 \rangle} \cdot \omega_1 , \omega_1 \right\rangle$

$$= \langle v_2, \omega_1 \rangle - \frac{\langle v_2, \omega_1 \rangle}{\langle \omega_1, \omega_1 \rangle} \langle \omega_1, \omega_1 \rangle$$

$$= \langle v_2, \omega_1 \rangle - \langle v_2, \omega_1 \rangle$$

$$= 0.$$

$\therefore \langle \omega_2, \omega_1 \rangle = 0.$

also $v_2 = \omega_2 + \frac{\langle v_2, v_1 \rangle}{\langle v_1, v_1 \rangle} \cdot v_1$

$\therefore v_2 = \alpha_1 v_1 + \omega_2$ where $\alpha_1 = \frac{\langle v_2, w_1 \rangle}{\langle w_1, \omega_1 \rangle} = \frac{\langle v_2, v_1 \rangle}{\langle v_1, v_1 \rangle}$

$\alpha_1 \in F.$

$\because v_1 \neq 0$

$\Rightarrow \langle v_1, v_1 \rangle \neq 0.$